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ON SOME CONNECTIONS BETWEEN ZETA-ZEROS AND SQUARE-FREE DIVISORS OF AN INTEGER

KAZIMIERZ WIERTELAK*

Abstract: A relationship between square-free divisors of an integer and zeros of the Riemann zeta-function, which is more explicit than the classical formula, is presented and discussed.
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1. Introduction and statement of results

Let $\theta(n)$ denote the number of square-free divisors of n. Moreover, let $s(z)$ and $S(z)$ be functions holomorphic of the upper half-plane defined by (1.1) and (1.2) below. In this note the analytic character of them is considered. In particular we show that they admit analytic continuation to multivalued functions on $\mathbb{C}$. Moreover, $s(z)$ satisfies certain functional equation (cf. Theorem 2 below).

In the case of simple zeros of the Riemann zeta-function, $s(z)$ and $S(z)$ are defined as follows:

$$s(z) = \lim_{n\to\infty} \sum_{\substack{\rho \\ 0<\Im\rho<T_n}} \frac{\zeta^2\left(\frac{\rho}{2}\right) e^{\frac{z\rho}{2}}}{2\zeta'(\rho)}$$

and

$$S(z) = \lim_{n\to\infty} \sum_{\substack{\rho \\ 0<\Im\rho<T_n}} \frac{\zeta^2\left(\frac{\rho}{2}\right) e^{\frac{z\rho}{2}}}{\rho\zeta'(\rho)},$$

where the summation is over non-trivial zeros ρ of $\zeta(s)$, and a suitably chosen sequence T_n yields an appropriate grouping of the zeros.

Similar investigations were performed by J. Kaczorowski [5] in connection with the distribution of primes in arithmetic progressions, and by K. Bartz [2] in connection with the Möbius μ-function. In our case singularities of $s(z)$ and $S(z)$ are more complicated when compared with singularities of the corresponding functions considered in [5] and [2].

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As an application of our results we give a new proof of the classical explicit formula for $\sum_{n\leqslant x}\theta(n)$.

In the general case, if $\zeta(s)$ has a multiple zero at $s=\rho$, the corresponding term in $s(z)$ and $S(z)$ must be replaced by the appropriate residue. Let k_ρ denote the multiplicity of a nontrivial zero ρ. Then the general definitions of $s(z)$ and $S(z)$ read as follows:

$$s(z)=\sum_{n=0}^{\infty}\sum_{T_n<\Im\rho<T_{n+1}}\frac{1}{2(k_\rho-1)!}\frac{d^{k_\rho-1}}{ds^{k_\rho-1}}\left[(s-\rho)^{k_\rho}\frac{e^{\frac{1}{2}sz}\zeta^2\left(\frac{s}{2}\right)}{\zeta(s)}\right]_{s=\rho} \tag{1.1}$$

$$=\sum_{n=0}^{\infty}s_n(z),$$

$$S(z)=\sum_{n=0}^{\infty}\sum_{T_n<\Im\rho<T_{n+1}}\frac{1}{(k_\rho-1)!}\frac{d^{k_\rho-1}}{ds^{k_\rho-1}}\left[\frac{(s-\rho)^{k_\rho}e^{\frac{1}{2}sz}\zeta^2\left(\frac{s}{2}\right)}{s\zeta(s)}\right]_{s=\rho} \tag{1.2}$$

$$=\sum_{n=0}^{\infty}S_n(z),$$

where $\Im z>0, T_0=14$, and $2^{n-1}K_0\leqslant T_n<2^nK_0$ $(n\geqslant 1, K_0$ being an absolute positive constant) denotes a suitable sequence of numbers (for the precise definition of T_n's see Section 2). It is easy to see that $s(z)$ and $S(z)$ are holomorphic for $\Im z>0$ (see Lemma 3).

Our principal aim is to describe analytic character of these functions. To this end let us introduce the following notation. For any two real numbers a and b we denote by $l(a,b)$ a simple and smooth curve $\tau:[0,1]\to\mathbb{C}$ such that $\tau(0)=a$, $\tau(1)=b$ and $0<\Im\tau(t)\leqslant 1$ for $t\in(0,1)$. Moreover,

$$\int_{l(a,b)}f(z)\,dz$$

for a meromorphic function f means that f is regular on the curve $l(a,b)$ and also regular in the open domain bounded by $l(a,b)$ and the interval $[a,b]$. Similar convention applies to integrals of type

$$\int_{\overline{l(a,b)}}f(z)\,dz.$$

For $z\in\mathbb{C}$ we write

$$h(z)=\int_{l(-\frac{1}{4},\frac{3}{2})}\frac{\zeta^2(s)}{\zeta(2s)}e^{zs}\,ds, \tag{1.3}$$

$$\overline{h}(z)=\int_{\overline{l(\frac{3}{2},-\frac{1}{4})}}\frac{\zeta^2(s)}{\zeta(2s)}e^{zs}\,ds. \tag{1.4}$$

Of course h and $\overline{h}$ are entire functions of z.

Theorem 1. *The function $s(z)$ is holomorphic on the upper half-plane $H = \{z \in \mathbb{C} : \Im z > 0\}$ and for $0 < \Im z < \pi$ we have*

$$2\pi i s(z) = \frac{i}{8} e^{-\frac{z}{2}} \sum_{m=1}^{\infty} \frac{a(m)}{m^{3/2}} \left(1 - \frac{1}{4me^z}\right)^{-\frac{3}{2}} - \tag{1.5}$$

$$- e^{\frac{3}{2}z} \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{3/2}(z - \log n)} + H(z) + h(z),$$

where $H(z)$ is holomorphic for $|\Im z| < \pi$, $h(z)$ is defined by (1.3), $a(m) = \sum_{l^2|m} l\mu(l) d\left(\frac{m}{l^2}\right)$ ($a(m) = 0$ iff $2^3 \parallel m$ or $3^2 \parallel m$), and the branch of the power function is chosen so that $\left(1 - \frac{1}{4me^z}\right)^{-\frac{3}{2}} \to 1$ for $\Re z \to \infty$.

Let D denote the complex plane with slits along half-lines $(-i\infty - \log(4m), -\log(4m)]$, where $m \in \mathbb{N}$, $2^3 \not\parallel m$ and $3^2 \not\parallel m$.

Theorem 2. *The function $s(z)$ can be continued analytically to a meromorphic function on D and satisfies the following functional equation*

$$s(z) + \overline{s(\overline{z})} = A(z), \tag{1.6}$$

where for $\Re z > -\log 4$

$$A(z) = -\frac{6}{\pi^2} e^z (z + 2\gamma - 2\frac{\zeta'}{\zeta}(2)) + \frac{e^z}{\pi^2} \sum_{k=1}^{\infty} \frac{e^{-2kz} \zeta^2(2k)}{\binom{4k-2}{2k-1} \zeta(4k-1)}. \tag{1.7}$$

The only singularities of $s(z)$ on D are the simple poles at the points $z = \log n$ ($n = 1, 2, \dots$) on the real axis with residues

$$\operatorname*{res}_{z=\log n} s(z) = -\frac{\theta(n)}{2\pi i}.$$

The function $A(z)$ can be continued analytically to a multivalued analytic function on $\mathbb{C}$ except for $z = -\log 4m \pm ik\pi, m \in \mathbb{N}, 2^3 \not\parallel m, 3^2 \not\parallel m, k = 0, 1, 2, \dots$, where there are polar branch points of order two.

Let us now describe analytic character of $S(z)$ using Theorems 1 and 2. It turns out that these results can be considered as a complex form of the well-known explicit formulae for $\sum_{n \leqslant x} \theta(n)$.

For $z \in H$ we have

$$S(z) = \int_{z+i\infty}^{z} s(u)\, du,$$

the path of integration being the half-line $u = z + iy, \infty \geqslant y \geqslant 0$ and $S(z)$ is defined by (1.2). Hence $S(z)$ can be continued analytically along every curve lying on D and not passing through the poles of $s(z)$. $S(z)$ becomes a multivalued function on D.

In fact, every pole of $s(z)$ becomes a logarithmic branch point for $S(z)$. In particular for $|z-\log n|<r_0$, $n=1,2,\dots$, $r_0>0$ sufficiently small, we can write

$$S(z)=-\frac{\theta(n)}{2\pi i}\log(z-\log n)+g(z), \tag{1.8}$$

where $g(z)$ is holomorphic in the disc $|z-\log n|<r_0$ and depends on the choice of the particular branch of S.

For a real x let us write

$$F(x)=\lim_{y\to 0^+}\Re S(x+iy). \tag{1.9}$$

It is obvious that this limit does exist for every x which is a regular point of S. For $z=\log n, n=1,2,\dots$, the limit exists as well by (1.8), since $\lim_{y\to 0^+}\operatorname{Arg}(iy)=\frac{\pi}{2}$. Furthermore, since for any $x_0>0$, $\lim_{y\to 0^+}\operatorname{Arg}(x_0+iy)=0$, $\lim_{y\to 0^+}\operatorname{Arg}(-x_0+iy)=\pi$ and $\lim_{y\to 0^+}\operatorname{Arg}(iy)=\frac{\pi}{2}$, we have

$$F(x)=\frac{1}{2}(F(x+0)+F(x-0)) \tag{1.10}$$

for every real x, $x\neq-\log 4m$, $m\in\mathbb{N}$, $2^3 \nparallel m$, $3^2 \nparallel m$.

Theorem 3. *For $x\neq\log n$, $x\neq-\log 4m$, $n=1,2,\dots$, $m\in\mathbb{N}$, $2^3\nparallel m$, $3^2\nparallel m$, the series $\sum_{k=0}^{\infty}S_k(x)$ is convergent to $S(x)$. The convergence is uniform in every closed interval not containing points of the form $\log n$ and $-\log 4m$. For $x=\log n$, $n=1,2,\dots$, the series $\sum_{k=0}^{\infty}\Re S_k(x)$ is convergent to $\lim_{y\to 0^+}\Re S(x+iy)=F(x)$.*

Theorem 4. *For $x>\frac{1}{4}$ we have*

$$\begin{aligned}\lim_{n\to\infty}\sum_{|\Im\rho|<T_n}&\frac{1}{(k_\rho-1)!}\frac{d^{k_\rho-1}}{ds^{k_\rho-1}}\left[(s-\rho)^{k_\rho}\frac{\zeta^2\left(\frac{s}{2}\right)}{s\zeta(s)}x^{\frac{s}{2}}\right]_{s=\rho} \\ =&R_0(x)-\frac{6}{\pi^2}x(\log x+2\gamma-1-2\frac{\zeta'}{\zeta}(2)) \\ &+\frac{1}{2}-\frac{1}{\pi^2 x}\sum_{k=0}^{\infty}\frac{\zeta^2(2k+2)x^{-2k}}{(2k+1)\binom{4k+2}{2k+1}\zeta(4k+3)}.\end{aligned} \tag{1.11}$$

where $R_0(x)$ is defined by (2.1) below.

2. Lemmas

Let for $x>0$

$$R(x)=\sum_{n\leqslant x}\theta(n),\qquad R_0(x)=\frac{1}{2}(R(x+0)+R(x-0)). \tag{2.1}$$

The symbols $\mu(n)$ and $d(n)$ denote as usual the Möbius function and the number of divisors of n respectively. Moreover, $\gamma=0,577\dots$ is the Euler constant.

Lemma 1 (see [4], Theorem 9.4 and [2] Lemma 1). *There exist positive constants* c_1, c_2 *and* t_0 *such that for* $T \geqslant t_0$, *between* T *and* $2T$ *there exists a* t *satisfying*

$$|\zeta(\sigma + it)|^{-1} \leqslant c_2 \log^{c_1} t \quad \text{for} \quad -1 \leqslant \sigma \leqslant 3. \tag{2.2}$$

Lemma 2. *For a sufficiently small positive* ε *we have*

$$\zeta(\sigma + it) = \begin{cases} O(t^{\frac{1}{3}-\varepsilon}) & \text{for } \frac{1}{4} \leqslant \sigma \leqslant \frac{3}{4}, \\ O(t^{\frac{1}{12}-\varepsilon}) & \text{for } \frac{3}{4} \leqslant \sigma \leqslant 3. \end{cases} \tag{2.3}$$

For the proof see [6] and [9].

We choose $K_0 \geqslant \max(t_0, 14)$ and let $T_n (n \geqslant 1)$, where

$$2^{n-1} K_0 \leqslant T_n < 2^n K_0$$

is such that

$$|\zeta(\sigma + iT_n)|^{-1} \leqslant c_2 \log^{c_1} T_n,$$

(cf. (2.2)). Of course $\zeta(s)$ has no zeros on the line $t = T_n$. Moreover, by (2.2), (2.3) and the functional equation of $\zeta(s)$ we have

$$\frac{\zeta^2(\sigma + \frac{T_n}{2} i)}{\zeta(2\sigma + T_n i)} = O(T_n^{\frac{2}{3}}) \tag{2.4}$$

uniformly for $-\frac{1}{4} \leqslant \sigma \leqslant \frac{3}{2}$.

Let us now consider uniformity of the convergence of $s(z)$ and $S(z)$ (see (1.1) and (1.2).

Lemma 3. *The series* $s(z)$ *and* $S(z)$ *are uniformly convergent for* $y = \Im z \geqslant \delta > 0$ *almost uniformly with respect to* $x = \Re z$.

This lemma follows from (2.4). The proof is similar to the proof of Lemma 2 in [2].

Lemma 4. *Let* $w_n = a_n + ib_n$, $n = 1, 2, 3, \ldots$, *denote complex numbers such that* $|a_n| \leqslant A$, $n \geqslant 1$, $b_1 \leqslant b_2 \leqslant \cdots$, $\lim_{n\to\infty} b_n = \infty$, $T_0' < b_1 < T_1' < T_2' < \ldots$, *denote real numbers such that* $\lim_{n\to\infty} T_n' = \infty$, h_n, $n \geqslant 1$ *be the largest natural number such that* $b_{h_n} \leqslant T_n'$ *and let* $f_n(z)$, $n = 1, 2, \ldots$ *be holomorphic functions for* $\Im z > -\delta$, $(\delta > 0)$. *Moreover, let the series*

$$f(z) = \sum_{n=1}^{\infty} \sum_{T_{n-1}' < b_k \leqslant T_n'} f_k(z) e^{w_k z},$$

converge for $y = \Im z > 0$ and satisfy the following two conditions

$$\left| \sum_{n=N+1}^{\infty} \sum_{T'_{n-1} < b_k \leqslant T'_n} f_k(z) e^{(w_k - w_{h_N})z} \right| = o(y^{-2}), N \to \infty, \tag{2.5}$$

for $y \to 0^+$ almost uniformly with respect to $x = \Re z$, and

$$\left| \sum_{n=1}^{N} \sum_{T'_{n-1} < b_k \leqslant T'_n} f_k(z) e^{(w_k - w_{h_N})z} \right| = o(y^{-2}), N \to \infty, \tag{2.6}$$

for $y \to 0^-$ also almost uniformly with respect to $x = \Re z$.

Then, if f is holomorphic at the boundary point $x_0 \in \mathbb{R}$, the series

$$\sum_{n=1}^{\infty} \sum_{T'_{n-1} < b_k \leqslant T'_n} f_k(x_0) e^{w_k x_0}$$

converges to $f(x_0)$. Moreover, the convergence is uniform on every compact real interval consisting of regular points of f only.

This result is a generalization of the classical theorem of M. Riesz [8]. The proof is similar to the proof of Theorem 4.2 in [5].

Lemma 5. *Let f be such as in Lemma 4 and let $b_{h_N} > T'_N - C$ where C is an absolute constant. Suppose that for certain $x_0 \in \mathbb{R}$ we have*

$$f(z) = g \log(z - x_0) + h_1(z)$$

for $|z - x_0| < r_0$, $\Im z > 0$, where g is a complex number and $h_1(z)$ is holomorphic in the whole disc $|z - x_0| < r_0$. Then for N tending to infinity

$$\sum_{n=1}^{N} \sum_{T'_{n-1} < b_k \leqslant T'_n} f_k(x_0) e^{w_k x_0} = -g \log T_N - g\gamma + h_1(x_0) + g\frac{\pi i}{2} + o(1).$$

The proof is similar to the proof of Theorem 4.3 in [5].

Corollary 1. *Let f be as in Lemma 5 and $\Re g = 0$. Then*

$$\lim_{y \to 0^+} \Re f(x_0 + iy) = \lim_{N \to \infty} \Re \sum_{n=1}^{\infty} \sum_{T'_{n-1} < b_k \leqslant T'_n} f_k(x_0) e^{w_k x_0}. \tag{2.7}$$

3. Proof of Theorem 1

Let us define the half-lines

L_1: the half-line: $s = -\frac{1}{4} + it,\ \infty > t \geqslant 0$,

L_2: the half-line: $s = \frac{3}{2} + it,\ 0 \leqslant t < \infty$,

$\overline{L}_1$, $\overline{L}_2$: half lines symmetrical upon the real axis to L_1 and L_2 respectively.

For $z \in H$ we have by (2.4)

$$2\pi i s(z) = \mathrm{s}_1(z) + \mathrm{s}_2(z) + h(z), \tag{3.1}$$

where

$$\mathrm{s}_1(z) = \int_{L_1} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} ds, \tag{3.2}$$

$$\mathrm{s}_2(z) = \int_{L_2} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} ds, \tag{3.3}$$

and h is defined by (1.3).

Let us consider s_1 first. From the functional equation for $\zeta(s)$ we get

$$\frac{\zeta^2(s)}{\zeta(2s)} = \frac{\sin^2 \pi s - 2\sin^2 \frac{\pi s}{2}}{\pi^2}\Gamma^2(1-s)\Gamma(2s)\frac{\zeta^2(1-s)}{\zeta(1-2s)}.$$

Hence we can split the integral (3.2) into four integrals

$$\mathrm{s}_1(z) = \mathrm{s}_{11}(z) + \mathrm{s}_{12}(z) + \mathrm{s}_{13}(z) + \mathrm{s}_{14}(z), \tag{3.4}$$

where

$$\mathrm{s}_{11}(z) = -\frac{2}{\pi^2}\int_{L_1} \sin^2\frac{\pi s}{2}\Gamma^2(1-s)\Gamma(2s)\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{zs}ds, \tag{3.5}$$

$$\mathrm{s}_{12}(z) = -\frac{i}{4\pi^{3/2}}\int_{L_1} \Gamma(1-s)\Gamma(s+\frac{1}{2})\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z+\pi i+\log 4)}ds,$$

$$\mathrm{s}_{13}(z) = \frac{i}{4\pi^{3/2}}\int_{\overline{L}_1} \Gamma(1-s)\Gamma(s+\frac{1}{2})\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z-\pi i+\log 4)}ds,$$

$$\mathrm{s}_{14}(z) = \frac{i}{4\pi^{3/2}}\int_{L_1-\overline{L}_1} \Gamma(1-s)\Gamma(s+\frac{1}{2})\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z-\pi i+\log 4)}ds.$$

Since $\Gamma(s) = O(|t|^{\sigma-\frac{1}{2}}\exp(-\frac{\pi}{2}|t|))$ $(s = \sigma + it), \mathrm{s}_{11}(z)$ is regular for $\Im z > -\pi, \mathrm{s}_{12}(z)$ for $\Im z > -2\pi, \mathrm{s}_{13}(z)$ for $\Im z < 2\pi$ and $\mathrm{s}_{14}(z)$ is regular for $O < \Im z < 2\pi$.

Suppose $z = x + iy, 0 < y < 2\pi, x > -2\log 2$. Applying Cauchy integral theorem we obtain

$$\mathrm{s}_{14}(z) = \frac{1}{4\pi^{3/2}i}\cdot 2\pi i \sum_{w} \operatorname*{Res}_{s=w}\Gamma(1-s)\Gamma(1+\frac{1}{2})\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z-\pi i+\log 4)},$$

where the summation is taken over all singularities of $\Gamma(s+\frac{1}{2})$ in the interval $(-\infty, -\frac{1}{2})$. Hence

$$\begin{aligned} \mathrm{s}_{14}(z) &= ie^{\frac{z}{2}} \sum_{k=1}^{\infty} \frac{(2k-1)!!}{8^k(k-1)!} \frac{\zeta^2(\frac{1}{2}+k)}{\zeta(2k)} e^{-kz} \\ &= \frac{ie^{-\frac{z}{2}}}{8} \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \frac{\mu(l)d(n)}{n^{3/2}l^2} \sum_{k=0}^{\infty} \binom{-\frac{3}{2}}{k} \left(\frac{-1}{4nl^2e^z}\right)^k \\ &= \frac{ie^{-\frac{z}{2}}}{8} \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} \frac{\mu(l)d(n)}{n^{3/2}l^2} \left(1-\frac{1}{4nl^2e^z}\right)^{-\frac{3}{2}} \\ &= \frac{ie^{-\frac{z}{2}}}{8} \sum_{m=1}^{\infty} \frac{a(m)}{m^{3/2}} \left(1-\frac{1}{4me^z}\right)^{-\frac{3}{2}}, \end{aligned} \tag{3.6}$$

which gives analytic continuation of $\mathrm{s}_{14}(z)$ to $z \in D$.

To compute $\mathrm{s}_2(z)$ it is enough to apply the definition $\zeta(s)$ in the half-plane $\Re s > 1$. Indeed, we have

$$\mathrm{s}_2(z) = \sum_{n=1}^{\infty} \theta(n) \int_{L_2} e^{s(z-\log n)} ds = e^{\frac{3}{2}z} \sum_{n=1}^{\infty} \frac{\theta(n)}{n^{3/2}(z-\log n)}. \tag{3.7}$$

Collecting (3.1) – (3.7) we get (1.5) and Theorem 1 follows.

4. Proof of Theorem 2

Let us consider the function

$$\overline{s}(z) = \lim_{n\to\infty} \sum_{-T_n<\Im\rho<0} \frac{1}{2(k_\rho-1)!} \frac{d^{k_\rho-1}}{ds^{k_\rho-1}} \left[(s-\rho)^{k_\rho} \frac{e^{\frac{1}{2}sz}\zeta^2(\frac{s}{2})}{\zeta(s)}\right]_{s=\rho} \tag{4.1}$$

defined for $z \in \overline{H} = \{z \in \mathbb{C} : \Im z < 0\}$. We have

$$2\pi i\overline{s}(z) = \overline{\mathrm{s}}_1(z) + \overline{\mathrm{s}}_2(z) + \overline{h}(z), \tag{4.2}$$

where

$$\overline{\mathrm{s}}_1(z) = -\int_{\overline{L}_1} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} ds, \tag{4.3}$$

$$\overline{\mathrm{s}}_2(z) = -\int_{\overline{L}_2} \frac{\zeta^2(s)}{\zeta(2s)} e^{zs} ds, \tag{4.4}$$

and $\overline{h}$ is defined by (1.4).

Expanding $\frac{\zeta^2(s)}{\zeta(2s)}$ in (4.4) into Dirichlet series and using (3.7) as the definition of $\overline{\mathrm{s}}_2(z)$ for $\Im z < 0$, it can easily be seen that

$$\overline{\mathrm{s}}_2(z) = -\mathrm{s}_2(z). \tag{4.5}$$

Let us consider $\overline{\mathrm{s}}_1$ next. We have

$$\overline{\mathrm{s}}_1(z) = \overline{\mathrm{s}}_{11}(z) + \overline{\mathrm{s}}_{12}(z) + \overline{\mathrm{s}}_{13}(z) + \overline{\mathrm{s}}_{14}(z), \tag{4.6}$$

where

$$\begin{aligned}
\overline{\mathrm{s}}_{11}(z) &= \frac{2}{\pi^2}\int_{\overline{L}_1} \sin^2\frac{\pi}{2}s\Gamma^2(1-s)\Gamma(2s)\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{zs}ds, \qquad (4.7)\\
\overline{\mathrm{s}}_{12}(z) &= \frac{i}{4\pi^{3/2}}\int_{L_1}\Gamma(1-s)\Gamma\left(s+\frac{1}{2}\right)\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z+\pi i+\log 4)}ds,\\
\overline{\mathrm{s}}_{13}(z) &= -\frac{i}{4\pi^{3/2}}\int_{\overline{L}_1}\Gamma(1-s)\Gamma\left(s+\frac{1}{2}\right)\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z-\pi i+\log 4)}ds,\\
\overline{\mathrm{s}}_{14}(z) &= -\frac{i}{4\pi^{3/2}}\int_{L_1-\overline{L}_1}\Gamma(1-s)\Gamma\left(s+\frac{1}{2}\right)\frac{\zeta^2(1-s)}{\zeta(1-2s)}e^{s(z+\pi i+\log 4)}ds
\end{aligned}$$

and $\overline{\mathrm{s}}_{11}(z)$ is regular for $y<\pi, \overline{\mathrm{s}}_{12}(z)$ for $y>-2\pi, \overline{\mathrm{s}}_{13}(z)$ for $y<2\pi$, and $\overline{\mathrm{s}}_{14}(z)$ is regular for $-2\pi<y<0$. Hence for $-\pi<\Im z<\pi$ we have

$$\overline{\mathrm{s}}_{12}(z) = -\mathrm{s}_{12}(z), \overline{\mathrm{s}}_{13}(z) = -\mathrm{s}_{13}(z). \tag{4.8}$$

Similarly as before using the Cauchy integral theorem, we get for $-2\pi<\Im z<0$, $\Re z > -2\log 2$

$$\overline{\mathrm{s}}_{14}(z) = \mathrm{s}_{14}(z). \tag{4.9}$$

Finally for $|y|<\pi, z\in D$ by (3.1), (3.4), (4.2), (4.5), (4.6), (4.8) and (4.9) we obtain

$$s(z)+\overline{s}(z) = \frac{1}{2\pi i}(h(z)+\overline{h}(z)) + \frac{1}{2\pi i}(\mathrm{s}_{11}(z)+\overline{\mathrm{s}}_{11}(z)) + \frac{1}{\pi i}\mathrm{s}_{14}(z). \tag{4.10}$$

Moreover, by the theorem of residues, using (1.3) and (1.4) we have for all z

$$h(z)+\overline{h}(z) = -2\pi i\mathrm{Res}_{s=1}\frac{\zeta^2(s)}{\zeta(2s)}e^{zs} = -2\pi i\frac{e^z}{\zeta(2)}(z+2\gamma-2\frac{\zeta'}{\zeta}(2)). \tag{4.11}$$

Thus for $z\in D$, $|y|<\pi$ by (3.6) and (4.11) we have

$$\begin{aligned}
s(z)+\overline{s}(z) =& \frac{e^z}{\zeta(2)}(z+2\gamma-2\frac{\zeta'}{\zeta}(2)) + \frac{1}{2\pi i}(\mathrm{s}_{11}(z)+\overline{\mathrm{s}}_{11}(z)) \qquad (4.12)\\
&+\frac{e^{-\frac{z}{2}}}{8\pi}\sum_{m=1}^{\infty}\frac{a(m)}{m^{3/2}}\left(1-\frac{1}{4me^z}\right)^{-\frac{3}{2}},
\end{aligned}$$

where $\mathrm{s}_{11}(z)+\overline{\mathrm{s}}_{11}(z)$ is holomorphic for $|y|<\pi$.

Suppose $z = x + iy, |y| < \pi, x > -2\log 2$. Applying the Cauchy integral theorem we obtain

$$\mathrm{s}_{11}(z) + \bar{\mathrm{s}}_{11}(z) = \frac{4i}{\pi}\sum_w \mathrm{Res}_{s=w} \sin^2\frac{\pi}{2}s\Gamma^2(1-s)\Gamma(2s)\frac{\zeta^2(1-s)}{\zeta(1-2s)}\mathrm{e}^{zs},$$

where the summation is taken over all singularities of $\Gamma(2s)$ lying on the half-line $(-\infty, -\frac{1}{2}]$. Therefore

$$\begin{aligned}\frac{1}{2\pi i}(\mathrm{s}_{11}(z) + \bar{\mathrm{s}}_{11}(z)) = &\frac{\mathrm{e}^z}{\pi^2}\sum_{k=1}^{\infty}\frac{\zeta^2(2k)}{\binom{4k-2}{2k-1}}\frac{\mathrm{e}^{-2kz}}{\zeta(4k-1)} \\ &- \frac{1}{8\pi}\mathrm{e}^{-\frac{z}{2}}\sum_{m=1}^{\infty}\frac{a(m)}{m^{3/2}}\left(1-\frac{1}{4m\mathrm{e}^z}\right)^{-\frac{3}{2}}.\end{aligned} \tag{4.13}$$

Collecting (4.12) and (4.13) we have

$$\begin{aligned}s(z) + \bar{s}(z) &= -\frac{\mathrm{e}^z}{\zeta(2)}(z + 2\gamma - 2\frac{\zeta'}{\zeta}(2)) + \frac{\mathrm{e}^z}{\pi^2}\sum_{k=1}^{\infty}\frac{\zeta^2(2k)\mathrm{e}^{-2kz}}{\binom{4k-2}{2k-1}\zeta(4k-1)} \\ &= A(z).\end{aligned} \tag{4.14}$$

We write

$$\sum_{k=1}^{\infty}\frac{\zeta^2(2k)\mathrm{e}^{-2kz}}{\binom{4k-2}{2k-1}\zeta(4k-1)} = B(z).$$

The function $B(z)$ is holomorphic and periodic, with period πi on the half-plane $\Re z > -2\log 2$. Hence from (4.12) and (4.14) the function $A(z)$ can be continued analytically to a multivalued analytic function on the whole complex plane $\mathbb{C}$ except for $z = -\log 4m \pm ik\pi$, $k = 0, 1, 2, \ldots$, $m \in \mathbb{N}$, $2^3 \not\| m$, $3^2 \not\| m$, where there are polar branch points of order two.

If ρ is a complex zero of $\zeta(s)$ then so is $\bar{\rho}$. Hence for $z \in H$ we get $s(z) = \overline{s(\bar{z})}$. Next using (4.14) we have (1.6) and the function $s(z)$ can be continued analytically to a meromorphic function on D.

5. Proof of Theorem 3

Let us number the complex zeros of $\zeta(s)$ lying on H according to increasing imaginary parts: $\rho_1, \rho_2, \rho_3, \ldots$ and in case of equal imaginary parts according to increasing real parts.

Let $\rho_{h_N} = \sigma_{h_N} + it_{h_N}$ be the last zero before the line $T = T_N$.

First we verify condition (2.5) of Lemma 4. Let us define the contour C_n consisting of the following four parts:

C_n^1: the line segment: $s=\sigma+i\frac{T_{n-1}}{2}$, $-\frac{1}{4}\leqslant\sigma\leqslant\frac{3}{2}$,
C_n^2: the line segment: $s=\frac{3}{2}+it$, $\frac{T_{n-1}}{2}\leqslant t\leqslant\frac{T_n}{2}$,
C_n^3: the line segment: $s=\sigma+i\frac{T_n}{2}$, $\frac{3}{2}\geqslant\sigma\geqslant-\frac{1}{4}$,
C_n^4: the line segment: $s=-\frac{1}{4}+it$, $\frac{T_n}{2}\geqslant t\geqslant\frac{T_{n-1}}{2}$.
By the Cauchy integral formula using estimate (2.4) we get

$$\left|\sum_{n=N+1}^{\infty}\sum_{T_{n-1}<\Im\rho<T_n}\frac{1}{(k_\rho-1)!}\frac{d^{k_\rho-1}}{ds^{k_\rho-1}}\left[\frac{(s-\rho)^{k_\rho}e^{\frac{1}{2}(s-\rho_{h_N})z}\zeta^2(\frac{s}{2})}{s\zeta(s)}\right]_{s=\rho}\right| \tag{5.1}$$
$$=\left|\frac{1}{2\pi i}\sum_{n=N+1}^{\infty}\int_{C_n}\frac{\zeta^2(s)}{s\zeta(2s)}e^{\frac{1}{2}(2s-\rho_{h_N})z}ds\right|$$
$$\ll\frac{e^{\frac{3}{2}|x|}}{y}\left(\frac{1}{2^{N/3}}+\frac{1}{y}\sum_{n=N+3}^{\infty}2^{-\frac{4}{3}n}\right)=\underset{N\to\infty}{o}(y^{-2})$$

for $y\to0^+$ almost uniformly with respect to x.
Similarly one can prove that

$$\left|\sum_{n=1}^{N}\sum_{T_{n-1}<\Im\rho<T_n}\frac{1}{(k_\rho-1)!}\frac{d^{k_\rho-1}}{ds^{k_\rho-1}}\left[\frac{(s-\rho)^{k_\rho}e^{\frac{1}{2}(s-\rho_{h_N})z}\zeta^2(\frac{s}{2})}{s\zeta(s)}\right]_{s=\rho}\right| \tag{5.2}$$
$$\ll\frac{e^{\frac{3}{2}|x|}}{|y|^2}\left(\sum_{n=1}^{N-3}\frac{1}{2^{N-3}2^{\frac{n-2}{3}}}+\frac{1}{2^{\frac{N-4}{3}}}\right)=o(|y|^{-2})\qquad(N\to\infty)$$

for $y\to0^-$ almost uniformly with resect to $x=\Re z$. Hence by Lemma 4 and Theorem 1 the series $\sum_{n=0}^{\infty}S_n(x)$ converges to $S(x)$ for $x\neq\log n$, $x\neq-\log 4m$, $n=1,2,\dots$, $m\in\mathbb{N}$, $2^3\not\|m$, $3^2\not\|m$.

The second part of Theorem 3 follows from (1.8) and Corollary of Lemma 5. Therefore Theorem 3 is proved.

6. Proof of Theorem 4

Suppose first that $x>\frac{1}{4}$ and $x\notin\mathbb{N}$, so that $s(z)$ is regular at $z=\log x$. Moreover, let $-\log x\leqslant a<2\log 2$. We have

$$S(\log x)=S(-a)+\int_l s(z)dz,$$

where $l=l(-a,\log x)$.

By the theorem of residues and Theorem 1 we obtain

$$\int_l s(z)dz-\int_{\bar{l}}s(z)dz=-2\pi i\sum_{n\leqslant x}\mathrm{Res}_{z=\log n}s(z)=R_0(x). \tag{6.1}$$

Moreover, using the functional equation (1.6) we get

$$\int_{\bar{l}} s(z)dz = \int_{l} s(\bar{z})d\bar{z} = \int_{l} (-\overline{s(z)} + \overline{A(z)})d\bar{z} \tag{6.2}$$
$$= -\overline{\int_{l} s(z)dz} + \int_{-a}^{\log x} A(t)dt.$$

Combining the above equalities we arrive at

$$2\Re S(\log x) - 2\Re S(-a) = R_0(x) + \int_{-a}^{\log x} A(t)\, dt \tag{6.3}$$
$$= R_0(x) - \frac{6x}{\pi^2}(\log x - 1 + 2\gamma - 2\frac{\zeta'}{\zeta}(2))$$
$$- \frac{1}{\pi^2}\sum_{k=1}^{\infty} \frac{\zeta^2(2k)x^{1-2k}}{(2k-1)\binom{4k-2}{2k-1}\zeta(4k-1)} - \frac{6e^{-a}}{\pi^2}(a + 1 - 2\gamma + 2\frac{\zeta'}{\zeta}(2))$$
$$+ \frac{1}{\pi^2}\sum_{k=1}^{\infty} \frac{\zeta^2(2k)e^{(2k-1)a}}{(2k-1)\binom{4k-2}{2k-1}\zeta(4k-1)}.$$

Hence, by Theorem 3 we get

$$2F(\log x) - 2F(-a) = R_0(x) - \frac{6x}{\pi^2}(\log x + 2\gamma - 1 - 2\frac{\zeta'}{\zeta}(2)) \tag{6.4}$$
$$- \frac{6}{\pi^2}e^{-a}(a + 1 - 2\gamma + 2\frac{\zeta'}{\zeta}(2))$$
$$- \frac{1}{\pi^2}\sum_{k=1}^{\infty} \frac{\zeta^2(2k)(x^{1-2k} - e^{-a(1-2k)})}{(2k-1)\binom{4k-2}{2k-1}\zeta(4k-1)}.$$

Let N be a positive integer and let $C_{N,n}$ denote the rectangle with vertices $-N + \frac{1}{2} - i\frac{T_n}{2}$, $\frac{3}{2} - i\frac{T_n}{2}$, $\frac{3}{2} + i\frac{T_n}{2}$ and $-N + \frac{1}{2} + i\frac{T_n}{2}$. Then for $\frac{1}{4} < y < 1$, we have

$$\lim_{\substack{N\to\infty \\ n\to\infty}} \int_{C_{N,n}} \frac{y^s\zeta^2(s)}{s\zeta(2s)} = 0$$

and by the theorem of residues

$$\mathrm{Res}_{s=0}\left(\frac{y^s\zeta^2(s)}{s\zeta(2s)}\right) + \mathrm{Res}_{s=1}\left(\frac{y^s\zeta^2(s)}{s\zeta(2s)}\right) + \lim_{n\to\infty} \sum_{\substack{\rho \\ |\Im\rho|<T_n}} \mathrm{Res}_{s=\rho}\left(\frac{y^s\zeta^2(s)}{s\zeta(2s)}\right) \tag{6.5}$$
$$+ \lim_{N\to\infty} \sum_{k=1}^{\infty} \mathrm{Res}_{s=-k} \frac{y^s\zeta^2(s)}{s\zeta(2s)} = 0.$$

Therefore

$$-\frac{1}{2}+\frac{y}{\zeta(2)}(\log y-1+2\gamma-2\frac{\zeta'}{\zeta}(2)) \\ +\frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{y^{1-2k}\zeta^2(2k)}{(2k-1)\binom{4k-2}{2k-1}\zeta(4k-1)}+2F(\log y)=0. \tag{6.6}$$

Hence, from (6.4) and (6.6) for $y=e^{-a}$ we have (1.11).

Now, let x be a positive integer, then $\log x$ is not a regular point of $s(z)$. From Theorem 3 we obtain

$$\begin{aligned} 2F(\log x) =& F(x+0)+F(x-0)=\frac{R(x+0)+R(x-0)}{2} \\ &-\frac{6x}{\pi^2}(\log x+2\gamma-1-2\frac{\zeta'}{\zeta}(2))+\frac{1}{2} \\ &-\frac{1}{\pi^2 x}\sum_{k=0}^{\infty}\frac{\zeta^2(2k+2)}{x^{2k}(2k+1)\binom{4k+2}{2k+1}\zeta(4k+3)}, \end{aligned}$$

and the proof is complete.

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Address: Adam Mickiewicz University, Faculty of Mathematics and Computer Science, 61-614 Poznań, Poland
E-mail: `wiertela@amu.edu.pl`

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