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# FUNDAMENTAL PROPERTIES OF SYMMETRIC SQUARE L-FUNCTIONS-II

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Dedicated with deep regards
to Professor Aleksandar Ivić.

**Abstract:** In this paper, we study certain density theorems of the zeros of the symmetric square $L$-functions (attached to a holomorphic cuspform defined over the full modular group) to the right of the critical line. We also obtain an analogous density theorem of Halasz and Turan for the zeros of the symmetric square $L$-functions.
**Keywords:** Symmetric square $L$-functions, zero-density theorems, mean-square bounds, growth conditions.

## 1. Introduction

A remarkable result of Selberg (see [58]) says that a positive proportion of zeros of the Riemann zeta-function are on the critical line. Similar results were obtained by Hafner in the case of L-functions attached to cusp forms which are Hecke eigenforms (see [12]). Another important problem is studying the growth of the $L$-functions under consideration. In this connection, in a celebrated paper [28], Iwaniec and Sarnak proved certain growth estimates for eigenfunctions of certain arithmetic surfaces which break beautifully the bound usually being obtained by convexity arguments. This raises the question of proving non-trivial (breaking the usual convexity bounds) growth estimates for general $L$-functions. Of course, this is closely related to the problem Lindelöf hypothesis. Also we should point out here other important works by Iwaniec, Duke, Friedlander and Iwaniec (see [27], [9], and [10]) which establish how one can break the convexity bounds in different aspects namely $Q$ and $r$ for certain automorphic L-functions of certain degree. For an excellent exposition of these results and about further comments we refer to [57]. Sometimes the question of studying the difference between consecutive zeros on the critical line were considered by various mathematicians (for example see [1], [5], [30], [31] and [54]). In [61], Shimura proved that the completed

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symmetric square $L$-functions can be continued analytically as an entire function on the whole complex plane by establishing the functional equation.In the recent times establishing the analytic continuation and the functional equation for various symmetric power L-functions is of much interest (see [60]).
We always mean $s = \sigma + it$, $z = x + iy$. Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a holomorphic cusp form of even integral weight $k$ defined over the full modular group $SL(2, \mathbb{Z})$. We assume that $a_n$ are eigen - values of all the Hecke operators and $a_1 = 1$. Let $\alpha_p$ and $\beta_p$ be the complex numbers satisfying the equation

$$1 - a_p p^{-s} + p^{k-1-2s} = \left(1 - \alpha_p p^{-s}\right)\left(1 - \beta_p p^{-s}\right). \tag{1.1}$$

The Hecke L-function attached to $f$ is defined as

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s} \tag{1.2}$$

which is absolutely convergent in a certain half-plane and is continuable analytically as an entire function on the whole plane. For an arbitrary primitive Dirichlet character $\psi$, the symmetric square L- function attached to $f$ is defined as

$$\begin{aligned} D(s, f, \psi) &=: D(s) \\ &= \prod_p \left((1 - \psi(p)\alpha_p^2 p^{-s})(1 - \psi(p)\beta_p^2 p^{-s})(1 - \psi(p)p^{k-1-s})\right)^{-1} \\ &=: \sum_{n=1}^{\infty} a_{n^2} n^{-s}. \end{aligned} \tag{1.3}$$

According to the notation in [61], throughout this paper we assume that $\chi$ (a Dirichlet character modulo $M$) is the trivial character with $M = 1$ and $\psi$ (an arbitrary primitive Dirichlet character with conductor $r$) is also the trivial character with $r = 1$. Now, $D(s)$ converges absolutely in $\Re s > k$. Here the critical strip is $k - 1 \leqslant \sigma \leqslant k$ and the critical line is $\sigma = k - 1/2$. In fact,

$$D(s) =: \zeta(2s - 2k + 2)\left(\sum_{n=1}^{\infty} a^*_{n^2} n^{-s}\right)$$

where $a^*_{n^2}$ are multiplicative and we also note that from Deligne's work (see [7] and [8]), they satisfy the inequality

$$|a^*_{n^2}| \leqslant d(n^2) n^{k-1}.$$

From our definition (1.3), it follows that

$$b_n =: a_{n^2} = \sum_{l^2 m = n} l^{2k-2} a^*_{m^2}$$

and $b_n$ are multiplicative. It is not difficult to see that (for $j \geqslant 5$)

$$|b_{p^j}| \leqslant p^{j(k-1)} \sum_{a=0}^{j} (2a+1) \sum_{2b+a=j} 1 \leqslant p^{j(k-1)}(j+1)^2 = p^{j(k-1)} \left(d\left(p^j\right)\right)^2$$

since there is utmost one solution $b$ to the equation $2b+a=j$ for every fixed $a$. For $0 \leqslant j \leqslant 4$, easy computation shows that the above inequality holds. This implies that (in our notation)

$$|a_{n^2}| \leqslant (d(n))^2 n^{k-1}. \tag{1.4}$$

In fact using the fundamental lemma 3.1 of this paper combined with lemma 3.2 of [32], it is not difficult to establish a reasonable zero - free region. It should me mentioned here that the mean values of derivatives of modular $L$-series had been studied earlier by Ram Murty and Kumar Murty in [50].
In recent times, the properties of Rankin-Selberg zeta-functions have been studied extensively by many authors (for example see [24], [25] and [34]). After normalizing the coefficients, the Rankin-Selberg zeta-function is defined as

$$Z(s) = \zeta(2s) \sum_{n=1}^{\infty} a_n^2 n^{1-k-s} = \sum_{n=1}^{\infty} c_n' n^{-s} \quad \text{(say)} \tag{1.5}$$

which is absolutely convergent in the half plane $\sigma > 1$, and it can be continued as a meromorphic function to the whole complex plane with a simple pole at $s=1$. It satisfies a nice functional equation (see [24]). For example, in [24], Ivic studied mean-value theorems for $Z(s)$ for a certain range of $\sigma$ and from his result it follows that

$$\int_0^T \left|Z(\frac{1}{2}+it)\right|^2 dt \ll T^{2+\epsilon} \tag{1.6}$$

for every $\epsilon > 0$. In [34], Matsumoto proved a refined upper bound for the mean-square of the absolute value of $Z(\sigma+it)$ (replacing $T^{\epsilon}$ into $(\log T)^C$) whenever $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$ and an asymptotic formula with certain error term whenever $\frac{3}{4} < \sigma \leqslant 1$ (see theorem 2 of [34]).
In part-I with the same title (see [55]), we improved the mean-square upper bound for Rankin-Selberg zeta-functions on any fixed line when $\frac{1}{2} \leqslant \sigma \leqslant \frac{3}{4}$ and improved the exponent of the error term in the asymptotic formula when $\frac{3}{4} < \sigma \leqslant 1$.
In this paper we consider the following fundamental question related to the symmetric square $L$-functions.
*Can we prove certain 'Density Theorems' for the zeros of $D(s)$ ?*
After Shimura's work (see [61]), the answers to the above important fundamental question do form the core part for further progress if any.

Let $N^*(\sigma, T)$ denote the number of zeros $\varrho = \beta + i\gamma$ of $D(s)$ with $\beta \geqslant \sigma$ and $|\gamma| \leqslant T$. From lemma (3.3), it is clear that $N^*(\sigma,T) \ll T \log T$ provided

$k-1/2 \leqslant \sigma \leqslant k-1/2+1/(\log T)$. For the Riemann zeta-function $\zeta(s)$, we know for example the familiar result of Ingham, namely

$$N(\sigma,T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^5 ,$$

in the case of $L$- functions an averaging result of Bombieri (see [2]) which states that when $T \leqslant Q$,

$$\sum_{q\leqslant Q} \sum_{\chi}{}^{*} N_\chi(\sigma,T) \ll TQ^{\frac{8(1-\sigma)}{3-2\sigma}} (\log Q)^{10} .$$

It is also known that

$$N(\sigma,T) \ll T^{\frac{12}{5}(1-\sigma)} (\log T)^{100} .$$

We also refer to [16] for sharp density results for the zeros of $\zeta(s)$ in certain range of $\sigma$. The following result is due to Montgomery (see [36]) which we state as

**Theorem A.** *For $T \geqslant 2$, let*

$$M(T) = \max_{\substack{2\leqslant t\leqslant T \\ \alpha \geqslant \frac{1}{2}}} |\zeta(\alpha+it)| .$$

*Then for $\frac{3}{4} \leqslant \sigma \leqslant 1$, we have*

$$N(\sigma,T) \ll \left\{ M(5T) (\log T)^6 \right\}^{\frac{8(1-\sigma)(3\sigma-2)}{(4\sigma-3)(2\sigma-1)}} (\log T)^{11} .$$

We prove some density theorems for $D(s)$ and discuss their further implications. In proving theorems 1.3 and 1.5, the central idea is to study how frequent certain Dirichlet polynomial being large. This main idea was developed and used first by Montgomery and later by many mathematicians (see [16], [17], [20], [26] and [29]). In this paper, we prove

**Theorem 1.1.** *For $\sigma \geqslant k - \frac{1}{2} + \frac{1}{\log T}$, we have*

$$N^*(\sigma,T) \ll T^{\frac{5(k-\sigma)}{(2k+1-2\sigma)}} (\log T)^C .$$

It is not hard to prove the following general theorem.

**Theorem 1.2.** *Assume the following conditions.*

*1. $G(s)$ has meromorphic continuation to the whole complex plane except for a simple pole at $s=1$.*

*2. $G(s)$ and $(G(s))^{-1}$ have the Dirichlet series representation namely*

$$G(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad ; \quad (G(s))^{-1} = \sum_{n=1}^{\infty} b_n n^{-s}$$

*in* $\sigma > 1$ *with the conditions* $a_1 = b_1 = 1$, $a_n$ *and* $b_n$ *are multiplicative and*

$$\max(|a_n|, |b_n|) \leqslant (\log n)^A$$

*for some fixed positive constant* $A$.

3. *For* $t \geqslant 10$, *we have*

$$G(1/2 + it) \ll t^C (\log t)^{C'}.$$

*If* $N_G(\sigma, T, 2T)$ *denotes the number of zeros* $\varrho = \beta + i\gamma$ *of* $G(s)$ *with* $\beta \geqslant \sigma (> \frac{1}{2}), T \leqslant \gamma \leqslant 2T$, *then*

$$N_G(\sigma, T, 2T) \ll T^{2(1+2C)(1-\sigma)} (\log T)^C.$$

**Remark.** Theorem 1.2 is better than the theorem 1.1 whenever $C < \frac{1}{8}$. Because, for $k - \frac{1}{2} \leqslant \sigma \leqslant k$, the inequality

$$\frac{5}{2k - 2\sigma + 1} > \frac{5}{2} > 2(1 + 2C)$$

holds only when $C < \frac{1}{8}$. However it should be pointed out here that we know only $C = \frac{3}{4}$ from (2.20).

**Theorem 1.3.** *(Halasz - Turan Type). Let* $\epsilon$ *be an arbitrarily small positive constant and* $\delta$ *be any sufficiently small positive constant but a fixed one. If* $D(k - 1/2 + it) \ll t^{\delta^2}$ *and* $\zeta(\frac{1}{2} + it) \ll t^{\delta^2}$ *for all* $t \geqslant t_0$ *then*

$$N^* \left(k - \frac{1}{4} + \delta^{1/2}, T\right) < T^{C\delta}$$

*where* $C$ *is a positive constant independent of the parameters* $\delta, \epsilon$ *and* $k$.

It is not hard to prove the following general theorem.

**Theorem 1.4.** *Assume the following conditions.*

1. $G(s)$ *has meromorphic continuation to the whole complex plane except for a simple pole at* $s = 1$.

2. $G(s)$ *and* $(G(s))^{-1}$ *have the Dirichlet series representation namely*

$$G(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad ; \quad (G(s))^{-1} = \sum_{n=1}^{\infty} b_n n^{-s}$$

*in* $\sigma > 1$ *with the conditions*

$$\sum_{n \leqslant x} |a_n|^2 \ll x (\log x)^A \quad ; \quad \sum_{n \leqslant x} |b_n|^2 \ll x (\log x)^B$$

*3.*

$$G(\frac{1}{2}+it) \ll t^{\delta^2} \quad ; \quad \zeta(\frac{1}{2}+it) \ll t^{\delta^2}$$

*for sufficiently small (but fixed) positive constant $\delta$. Then we have*

$$N_G\left(\frac{3}{4}+\delta^{1/2}, T, 2T\right) \ll T^{C\delta}.$$

**Theorem 1.5.** *Let*

$$M_1 = M_1\left(\frac{1}{2}, 2T\right) = \max_{\frac{T}{2}\leqslant t\leqslant 2T}\left|\zeta(\frac{1}{2}+it)\right|$$

*and*

$$M_2 = M_2\left(k-\frac{1}{2}, 2T\right) = \max_{\frac{T}{2}\leqslant t\leqslant 2T}\left|D(k-\frac{1}{2}+it)\right|.$$

*Then for $k-\frac{1}{4}<\sigma\leqslant k$, we have*

$$N^*(\sigma,T) \ll \left(M_2^2\left(M_1^{\frac{1}{2}}(\log T)^{2^{61}}\right)^{\frac{4}{4\sigma-4k+1}}(\log T)^{2^{31}}\right)^{\frac{2(k-\sigma)}{2\sigma-2k+1}}(\log T)^5.$$

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## 2. Notation and preliminaries

The letters $C$ and $A$ (with or without suffixes) denote effective positive constants unless it is specified. It need not be the same at every occurrence. Throughout the paper we assume $T \geqslant T_0$ where $T_0$ is a large positive constant. We write $f(x) \ll g(x)$ to mean $|f(x)| < C_1 g(x)$ (sometimes we denote this by the $O$ notation also). Let $s=\sigma+it$, and $w=u+iv$. The implied constants are all effective. In any fixed strip $a\leqslant\sigma\leqslant b$, as $t\to\infty$, we have

$$\Gamma(\sigma+it) = t^{\sigma+it-1/2}e^{-\pi/2-it+(i\pi/2)(\sigma-1/2)}\sqrt{2\pi}\left(1+O\left(1/t\right)\right). \tag{2.1}$$

Let

$$R(s) = \pi^{-3s/2}\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\Gamma(\frac{s-k+2}{2})D(s). \tag{2.2}$$

Then $D(s)$ satisfies the functional equation (see [61])

$$R(s) = R(2k-1-s) \tag{2.3}$$

Also we note that if

$$R_1(s) = \pi^{-\frac{(s-k+1)}{2}}\Gamma(\frac{s-k+1}{2})\zeta(s-k+1) \tag{2.4}$$

then $\zeta(s-k+1)$ satisfies the functional equation

$$R_1(s) = R_1(2k-1-s). \tag{2.5}$$

Therefore if $D_1(s) = \zeta(s-k+1)D(s)$, then $D_1(s)$ satisfies the functional equation

$$R(s)R_1(s) = R(2k-1-s)R_1(2k-1-s) \tag{2.6}$$

and we notice that $R(s)R_1(s)$ extends $D_1(s)$ as a meromorphic function to the whole plane except for a simple pole at $s = k$. We define

$$\xi(s) = -(s-k)(2k-1-s-k)R(s)R_1(s). \tag{2.7}$$

Note that

$$\xi(s) = \xi(2k-1-s). \tag{2.8}$$

We write

$$D(s) = \chi(s)D(2k-1-s) \tag{2.9}$$

where

$$\chi(s) = \pi^{\frac{-3(2k-1)}{2}+3s}\frac{\Gamma(\frac{2k-1-s}{2})\Gamma(\frac{2k-s}{2})\Gamma(\frac{k-s+1}{2})}{\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2})\Gamma(\frac{s-k+2}{2})}. \tag{2.10}$$

From (2.1) and (2.10), it follows that, for $a \leqslant \sigma \leqslant b$, we have as $t \to \infty$,

$$(\chi(s)) = C_2(k,\sigma)t^{\frac{1}{2}(6k-6\sigma-3)}(\frac{t}{2\pi e})^{-3it}\left(1+O\left(1/t\right)\right) \tag{2.11}$$

where $C_2$ is a certain constant depends only on $k$ and $\sigma$. We notice that for $\sigma > k$,

$$\frac{1}{D(s)} = \prod_p \left\{1 - \frac{(a_p^2 - p^{k-1})}{p^s} + \frac{(p^{2k-2} + p^{k-1}(a_p^2 - 2p^{k-1}))}{p^{2s}} - \frac{p^{3k-3}}{p^{3s}}\right\} \tag{2.12}$$

since

$$\alpha_p + \beta_p = a_p \quad ; \quad \alpha_p\beta_p = p^{k-1}.$$

Now we define a multiplicative function $\mu^*(n)$ in the following way.

$$\mu^*(n) = \begin{cases} 1 & \text{if } n = 1, \\ -(a_p^2 - p^{k-1}) & \text{if } n = p, \\ (p^{2k-2} + p^{k-1}(a_p^2 - 2p^{k-1})) & \text{if } n = p^2, \\ -p^{3k-3} & \text{if } n = p^3, \\ 0 & \text{if } n = p^a \text{ for any integer } a \geqslant 4 \end{cases} \tag{2.13}$$

and hence clearly

$$|\mu^*(n)| \leqslant (d(n))^3 n^{k-1} \tag{2.14}$$

and for $\sigma > k$,

$$\frac{1}{D(s)} = \sum_{n=1}^{\infty} \frac{\mu^*(n)}{n^s}. \tag{2.15}$$

Also we define $\mu_1^*$ in the following way. We note that

$$\frac{1}{\zeta(s-k+1)D(s)} = \prod_p \left\{ 1 - \frac{(a_p^2 - p^{k-1})}{p^s} + \frac{(p^{2k-2} + p^{k-1}(a_p^2 - 2p^{k-1}))}{p^{2s}} - \frac{p^{3k-3}}{p^{3s}} \right\} \cdot \left\{ 1 - \frac{p^{k-1}}{p^s} \right\} \tag{2.16}$$

and therefore define

$$\mu_1^*(n) = \begin{cases} 1 & \text{if } n = 1, \\ -a_p^2 & \text{if } n = p, \\ 2p^{k-1}(a_p^2 - p^{k-1})) & \text{if } n = p^2, \\ -p^{2k-2}a_p^2 & \text{if } n = p^3, \\ p^{4k-4} & \text{if } n = p^4, \\ 0 & \text{if } n = p^a \text{ for any integer } a \geqslant 5 \end{cases} \tag{2.17}$$

and hence clearly

$$|\mu_1^*(n)| \leqslant (d(n))^4 n^{k-1} \tag{2.18}$$

and for $\sigma > k$,

$$\frac{1}{\zeta(s-k+1)D(s)} = \sum_{n=1}^{\infty} \frac{\mu_1^*(n)}{n^s}. \tag{2.19}$$

From maximum-modulus principle and the functional equation, one has

$$D(\sigma + it) \ll |t|^{\frac{3}{2}(k-\sigma)} \log |t| \tag{2.20}$$

uniformly for $k - \frac{1}{2} \leqslant \sigma \leqslant k, |t| \geqslant 10$.

## 3. Some lemmas

**Lemma 3.1.** *(A Fundamental Identity) We have*

$$g(s) =: \sum_{n=1}^{\infty} a_n^2 n^{-s} = \frac{\zeta^2(s-k+1)}{\zeta(2s-2k+2)} \Psi(s)$$

*where*

$$\Psi(s) = \prod_p \left(1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2k-2-2s}\right)^{-1}.$$

**Proof.** See for example lemma 3.1 of [55]. For the sake of completeness, we reproduce the proof here.

We note that $a_n$ is multiplicative and satisfy the following equations

$$a_{p^\lambda} = a_p a_{p^{\lambda-1}} - p^{k-1} a_{p^{\lambda-2}} \tag{3.1}$$

and

$$p^{k-1} a_{p^{\lambda-3}} = -a_{p^{\lambda-1}} + a_p a_{p^{\lambda-2}}. \tag{3.2}$$

Therefore we can write first

$$g(s) = \prod_p \left\{ 1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} + \sum_{j=3}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right\}. \tag{3.3}$$

From these two equations (3.1) and (3.2), we observe that $(3.1)^2 - p^{k-1}(3.2)^2$ gives the relation

$$a_{p^\lambda}^2 - \{a_p^2 - p^{k-1}\} a_{p^{\lambda-1}}^2 + p^{k-1}\{a_p^2 - p^{k-1}\} a_{p^{\lambda-2}}^2 - p^{3(k-1)} a_{p^{\lambda-3}}^2 = 0. \tag{3.4}$$

Now, we write

$$\begin{aligned}
\sum_{j=3}^{\infty} \frac{a_{p^j}^2}{p^{js}} &= \sum_{j=3}^{\infty} \frac{(a_p^2 - p^{k-1}) a_{p^{j-1}}^2 - p^{k-1}(a_p^2 - p^{k-1}) a_{p^{j-2}}^2 + p^{3(k-1)} a_{p^{j-3}}^2}{p^{js}} \\
&= (a_p^2 - p^{k-1}) p^{-s} \left( \sum_{j=2}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right) - p^{k-1}(a_p^2 - p^{k-1}) p^{-2s} \left( \sum_{j=1}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right) \\
&\quad + p^{3(k-1)-3s} \left( \sum_{j=0}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right) \\
&= \left\{ (a_p^2 - p^{k-1}) p^{-s} - p^{k-1-2s}(a_p^2 - p^{k-1}) + p^{3(k-1)-3s} \right\} \left( \sum_{j=0}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right) \\
&\quad - (a_p^2 - p^{k-1}) p^{-s} - a_p^2 (a_p^2 - p^{k-1}) p^{-2s} + (a_p^2 - p^{k-1}) p^{k-1-2s}.
\end{aligned} \tag{3.5}$$

We also find that

$$\begin{aligned}
&1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} - (a_p^2 - p^{k-1}) p^{-s} \\
&- a_p^2 (a_p^2 - p^{k-1}) p^{-2s} + (a_p^2 - p^{k-1}) p^{k-1-2s}. = 1 + p^{k-1-s}.
\end{aligned}$$

Let

$$X = \left\{ 1 + a_p^2 p^{-s} + a_{p^2}^2 p^{-2s} + \sum_{j=3}^{\infty} \frac{a_{p^j}^2}{p^{js}} \right\}$$

and

$$Y = \left\{(a_p^2 - p^{k-1})p^{-s} - p^{k-1-2s}(a_p^2 - p^{k-1}) + p^{3(k-1)-3s}\right\}.$$

From (3.4), (3.5) and the above arguments, we observe that

$$X = XY + 1 + p^{k-1-s}, \tag{3.6}$$

with the above notion for $X$ and $Y$. Therefore we obtain

$$\begin{aligned} g(s) &= \prod_p \left( \frac{1+p^{k-1-s}}{1-(a_p^2 - p^{k-1})p^{-s} + p^{k-1-2s}(a_p^2 - p^{k-1}) - p^{3(k-1)-3s}} \right) \\ &= \prod_p \left\{ \frac{1-p^{k-1-s}}{1+p^{k-1-s}}(1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2(k-1)-2s}) \right\}^{-1} \\ &= \frac{\zeta^2(s-k+1)}{\zeta(2s-2k+2)}\Psi(s) \end{aligned} \tag{3.7}$$

where

$$\Psi(s) = \prod_p \left(1 + 2p^{k-1-s} - a_p^2 p^{-s} + p^{2(k-1)-2s}\right)^{-1}. \tag{3.8}$$

This proves the lemma. ∎

**Lemma 3.2.** *(Montgomery-Vaughan) If $h_n$ is an infinite sequence of complex numbers such that $\sum_{n=1}^{\infty} n|h_n|^2$ is convergent, then*

$$\int_T^{T+H} \left|\sum_{n=1}^{\infty} h_n n^{-it}\right|^2 dt = \sum_{n=1}^{\infty} |h_n|^2 (H + O(n)).$$

**Proof.** See for example lemma 3.3 of [39] or [46]. ∎

**Lemma 3.3.** *If $N^*(\sigma, T, T+1)$ denotes the number of zeros $\varrho = \beta + i\gamma$ of $D(s)$ with $\beta \geqslant \sigma, T \leqslant \gamma < T+1$, then*

$$N^*(k-1, T, T+1) \ll K_1 \log T.$$

**Proof.** We define

$$F_1(s) = \frac{D(s)}{\prod_{\varrho} \left(1 - \frac{s-s_0}{\varrho - s_0}\right)}$$

where $s_0 = k + i\gamma + \epsilon$ and $\varrho$ in the product runs over the zeros of $D(s)$ with $k-1 \leqslant \beta \leqslant k$ and $T < \gamma < T+1$. We note that

$$\begin{aligned} |F_1(s_0)| &= |D(s_0)| \\ &\geqslant 1 - \sum_{n=2}^{\infty} \frac{|a_{n^2}|}{n^{k+\epsilon}}, \\ &\geqslant 1 - \sum_{n=2}^{\infty} \frac{(d(n))^2 n^{k-1}}{n^{k+\epsilon}} \\ &\geqslant C \end{aligned}$$

where $C$ is a certain positive constant. This implies that

$$\begin{aligned} C &< |F_1(s_0)| \\ &< \max_{|s-s_0|=12} |F_1(s)| \\ &< \max_{|s-s_0|\leqslant 12} \frac{|D(s)|}{2^{N^*}} \\ &< \frac{T^C}{2^{N^*}} \end{aligned}$$

and hence we obtain the lemma.. ■

**Lemma 3.4.** *Let* $D(s) \sum_{n\leqslant X} \frac{\mu^*(n)}{n^s} - 1 = \sum_{n>X} \frac{c_n}{n^s}$, *then*

$$|c_n| \leqslant (d(n))^{30} n^{k-1}.$$

**Proof.** Since $a_{n^2}$ and $\mu^*$ are multiplicative functions, from the definition, $c_n$ is a multiplicative function. Therefore it is enough to prove the lemma on prime powers. From (2.14), we notice that (with $b_n = a_{n^2}$)

$$\begin{aligned} |c_{p^m}| &= \left| b_1\mu^*(p^m) + b_p\mu^*(p^{m-1}) + \cdots + b_{p^m}\mu^*(1) \right| \\ &\leqslant p^{m(k-1)} \left( \sum_{j=0}^{m} \left(d\left(p^j\right)\right)^2 \left(d\left(p^{m-j}\right)\right)^3 \right) \\ &\leqslant p^{m(k-1)} \left( \sum_{j=0}^{m} (j+1)^2 (m-j+1)^3 \right) \\ &\leqslant \left(d\left(p^m\right)\right)^6 p^{m(k-1)} \end{aligned}$$

which proves the lemma. ■

## 4. Proof of the theorems

**Proof of Theorem 1.1.** It is enough to prove the theorem for $T \leqslant t \leqslant 2T$. We divide the rectangle bounded by the lines with real parts $\sigma, 1$ and the imaginary parts $T, 2T$ into abutting smaller rectangles of width 1. We count the number of these smaller rectangles of width 1 which contain at least one zero and multiply by $\log T$ to get a bound for $N(\sigma, T, 2T)$. Define the function

$$F_2(s) = D(s) \sum_{n\leqslant T} \frac{\mu^*(n)}{n^s} - 1 = D(s)M_T(s) - 1 = \sum_{n\geqslant T} \frac{c'_n}{n^s} \text{ say.} \tag{4.1}$$

Let

$$G(s) = F_2(s) Y^{s-\varrho} e^{(s-\varrho)^2} \tag{4.2}$$

Now, select a set of zeros in each of the rectangles which contain a zero and consider the integral

$$\frac{1}{2\pi i} \int\limits_{R} \frac{G(s)}{s-\varrho} ds = \text{ the multiplicity of the zero } \varrho \tag{4.3}$$

(by Cauchy's residue theorem) where the integral being taken over the rectangle $R$ defined by $\varrho + x + iy$, $k - \frac{1}{2} \leqslant \beta + x \leqslant k$, $|\gamma + y| \leqslant (\log T)^2$. If $Y$ is chosen to satisfy $\log Y \ll \log T$, then the contributions from the horizontal sides of this rectangle $R$ is $O(T^{-10})$. Let us denote the vertical sides of $R$ by $V_1$ and $V_2$ so that we have

$$\begin{aligned} 1 &= O\left(\log T \left(\int\limits_{V_1} |F_2(s)| dt\right) Y^{k-\frac{1}{2}-\beta} + \log T \left(\int\limits_{V_2} |F_2(s)| dt\right) Y^{k-\beta}\right) \\ &= O\left(\log T \left(1 + \int\limits_{V_1} |F_2(s)| dt\right) Y^{k-\frac{1}{2}-\beta}\right. \\ &\quad \left. + \log T \left(T^{-10} + \int\limits_{V_2} |F_2(s)| dt\right) Y^{k-\beta}\right). \end{aligned} \tag{4.4}$$

We choose $Y$ such that

$$Y^{k-1/2-\beta}\left(1 + \int\limits_{V_1} |F_2(s)| dt\right) = Y^{k-\beta}\left(T^{-10} + \int\limits_{V_2} |F_2(s)| dt\right). \tag{4.5}$$

Let

$$J_1(\varrho) = \left(1 + \int\limits_{V_1} |F_2(s)| dt\right) \tag{4.6}$$

and

$$J_2(\varrho) = \left(T^{-10} + \int\limits_{V_2} |F_2(s)| dt\right). \tag{4.7}$$

Note that

$$Y = \left(\frac{J_1}{J_2}\right)^2 \geqslant \frac{1}{T^{-10} + T^C}; \quad Y \leqslant \frac{T^C}{T^{-10}}$$

so that the condition on $Y$ is satisfied. Hence we have

$$1 \leqslant 2C \log T \left(\frac{J_1}{J_2}\right)^{2(k-\beta)} J_2 \doteq 2C \log T J_1^{2(k-\beta)} J_2^{2\beta+1-2k}. \tag{4.8}$$

We notice that

$$\begin{aligned}\int_T^{2T} |M_T(k-1/2+1/\log T+it)|^2\,dt &= \sum_{n\leqslant T} \frac{|\mu^*(n)|^2}{n^{2(k-1/2+1/\log T)}}(T+O(n)) \\ &\leqslant \sum_{n\leqslant T} \frac{(d(n))^8 n^{2k-2}}{n^{2k-1}}(T+O(n)) \\ &\ll T(\log T)^C. \qquad (4.9)\end{aligned}$$

From (i) of theorem 4.1 of [55], we have

$$\int_T^{2T} |D(k-1/2+1/\log T+it)|^2\,dt \ll T^{3/2}(\log T)^C.$$

Therefore using Hölder's inequality, we find that

$$\int_T^{2T} |F_2(k-1/2+1/\log T+it)|\,dt \ll T^{5/4}(\log T)^C \qquad (4.10)$$

and using lemma 3.2 (Montgomery - Vaughan Theorem), we have

$$\begin{aligned}\int_T^{2T} |F_2(k+1/(\log T)+it)|^2\,dt &= \sum_{n\geqslant T} \frac{|c_n'|^2}{n^{2k+2/(\log T)}}(T+O(n)) \\ &\ll T\sum_{n\geqslant T}\frac{(d(n))^{30}}{n^{2+\frac{2}{\log T}}} + \sum_{n\geqslant T}\frac{(d(n))^{30}}{n^{1+\frac{2}{\log T}}} \\ &\ll (\log T)^C, \qquad (4.11)\end{aligned}$$

since $c_n' \ll d(n)^{30} n^{k-1}$ (from lemma 3.4). From (4.10) and (4.11) (using convexity arguments), it follows that

$$\sum_{\varrho} J_1(\varrho) < T^{5/4}(\log T)^C \quad ; \quad \sum_{\varrho} J_2(\varrho)^2 < (\log T)^C \qquad (4.12)$$

and so

$$|\{\varrho/J_1 \geqslant W_1\}| \leqslant \frac{T^{5/4}(\log T)^C}{W_1} \quad ; \quad |\{\varrho/J_2 \geqslant W_2\}| \leqslant \frac{(\log T)^C}{W_2^2}. \qquad (4.13)$$

Now we fix $W_1 = W_2^2 T^{5/4}$.Hence the total contribution from the above exception is

$$(\log T)^C\left\{\frac{T^{5/4}}{W_1} + \frac{1}{W_2^2}\right\}. \qquad (4.14)$$

From (4.4), for the remaining zeros, we have

$$\begin{aligned} 3/4 &\leqslant 2C(\log T)W_1^{2(k-\beta)}W_2^{2\beta+1-2k} \\ &= 2C(\log T)W_1^{2(k-\sigma)}W_1^{2(\sigma-\beta)}W_2^{2\sigma+1-2k}W_2^{2(\beta-\sigma)} \\ &= 2C(\log T)W_1^{2(k-\sigma)}W_2^{2\sigma+1-2k}\left(\frac{W_2}{W_1}\right)^{2(\beta-\sigma)} \\ &= 2C(\log T)W_1^{2(k-\sigma)}W_2^{2\sigma+1-2k}\left(\frac{1}{W_2T^{5/4}}\right)^{2(\beta-\sigma)}. \end{aligned} \tag{4.15}$$

Let us suppose that $W_2 > 1/T^{5/4}$ and so $\left(\frac{1}{W_2T^{5/4}}\right)^{2(\beta-\sigma)} < 1$. Therefore we get,

$$\begin{aligned} 3/4 &\leqslant 2C(\log T)W_1^{2(k-\sigma)}W_2^{2\sigma+1-2k} \\ &= 2C(\log T)\left(W_2^2T^{5/4}\right)^{2(k-\sigma)}W_2^{2\sigma+1-2k} \\ &= 2C(\log T)T^{5/2(k-\sigma)}W_2^{2k-2\sigma+1}. \end{aligned} \tag{4.16}$$

We choose

$$W_2 = \left(\frac{\log T}{4C}\right)^{-\frac{1}{2k-2\sigma+1}}T^{-\frac{5(k-\sigma)/2}{2k-2\sigma+1}}. \tag{4.17}$$

Clearly $W_2 > T^{-5/4}$. For this choice of $W_2$, (4.16) implies that $3/4 < 1/2$ which is absurd and this means that we should count only those zeros in (4.13). Hence we get,

$$N^*(\sigma, T, 2T) \ll \frac{(\log T)^C}{W_2^2} \ll T^{\frac{5(k-\sigma)}{2k-2\sigma+1}}(\log T)^C \tag{4.18}$$

which proves the theorem.

**Proof of theorem 1.2.** From the assumption 2 of the theorem it follows immediately that

$$\sum_{n\leqslant x}|a_n|^2 \ll x(\log x)^{2A} \quad ; \quad \sum_{n\leqslant x}|b_n|^2 \ll x(\log x)^{2A}. \tag{4.19}$$

We define

$$f_X(s) = G(s)\sum_{n<X}\frac{b_n}{n^s} - 1 = \sum_{n=1}^{\infty}\frac{C_X(n)}{n^s}. \tag{4.20}$$

We observe that

$$C_X(1) = 0 \;\; ; \;\; C_X(n) = 0 \text{ for } n < X \;\; ; \text{ and } |C_X(n)| \leqslant d(n)(\log n)^{2A}. \tag{4.21}$$

Therefore we have,

$$\sum_{n\leqslant x}|C_X(n)|^2 \ll x(\log x)^{4A+3} \tag{4.22}$$

and

$$\sum_{m<n\leqslant x} \frac{|C_X(m)C_X(n)|}{(mn)^{\frac{1}{2}}(\log(\frac{m}{n})} \ll x(\log x)^{4A+3}. \tag{4.23}$$

Now we follow the proof of theorem 9.18 in [62] and obtain the estimates

$$\int_0^T |f_X(1+\delta+it)|^2\,dt \ll (\frac{T}{X}+1)(\delta)^{-(4A+4)} \tag{4.24}$$

and

$$\int_0^T \left|f_X(\frac{1}{2}+it)\right|^2 dt \ll T^{2C}\,(T+X)\,(\log(T+2))^{2C'}\,(\log X)^{2A+1}. \tag{4.25}$$

Using convexity theorem and choosing $X = CT, \delta = (\log(T+X))^{-1}$, we obtain

$$\begin{aligned}\int_{\frac{T}{2}}^T |f_X(\sigma+it)|^2\,dt &\ll (T+X)T^{4C(1-\sigma)}X^{1-2\sigma}\,(\log(T+X))^{12(A+1)+2C'}\\ &\ll T^{(4C+2)(1-\sigma)}\,(\log T)^{12(A+1)+2C'}.\end{aligned}$$

This proves the theorem.

**Proof of theorem 1.3.** Let $\varrho = \beta + i\gamma, T \leqslant \gamma \leqslant 2T$ with $\beta > k - 1/4 + \delta^{1/2}$. We define (as before)

$$F(s) = D(s)\sum_{n\leqslant T^\delta} \frac{\mu^*(n)}{n^s} - 1 = D(s)M_{T^\delta}(s) - 1 = \sum_{n>T^\delta} \frac{c_n}{n^s} \text{ say.} \tag{4.26}$$

We notice that

$$c_n = 0 \text{ for } n \leqslant T^\delta; |c_n| \leqslant (d(n))^{30}n^{k-1}. \tag{4.27}$$

Now, from Mellin's tranform we have

$$\sum_{n>T^\delta} \frac{c_n}{n^s}e^{-\frac{n}{X}} = \frac{1}{2\pi i}\int_{\Re w=k+1} F(s+w)X^w\Gamma(w)dw. \tag{4.28}$$

We note that the truncated integral with $(\Re w = k+1, |v| \geqslant (\log T)^2)$ gives an error $o(1)$. Now, we move the line of integration of the remaining portion to the line $\Re w = -1/4$. By doing so, the residue contribution coming from the simple pole at $w = 0$ is $F(s)$. We fix throughout this proof $X = T^{20\delta}$. Therefore we obtain, (for $\sigma \geqslant k - 1/4, T \leqslant t \leqslant 2T$)

$$\begin{aligned}&\sum_{T^\delta\leqslant n\leqslant T^{21\delta}} \frac{c_n}{n^s}e^{-\frac{n}{X}} + \sum_{n>T^{21\delta}} \frac{c_n}{n^s}e^{-\frac{n}{X}} + o(1)\\ &= \frac{1}{2\pi i}\int_{\substack{\Re w=-1/4,\\ |v|\leqslant(\log T)^2}} F(s+w)X^w\Gamma(w)dw + F(s)\end{aligned} \tag{4.29}$$

Notice that (for $\sigma \geqslant k - 1/4$)

$$\begin{aligned}\sum_{n \geqslant X(\log X)^2} \frac{c_n}{n^s} e^{-\frac{n}{X}} &\ll \sum_{n \geqslant X(\log X)^2} e^{-\frac{n}{X}} \\ &\ll \frac{e^{-j/X}}{1 - e^{-1/X}} \\ &\ll X e^{\frac{1-j}{X}} \\ &= o(1) \quad \text{as} \quad X \to \infty \end{aligned} \tag{4.30}$$

since $j > [X(\log X)^2]$. Also we have

$$\frac{1}{2\pi i} \int\limits_{\substack{\Re w = -1/4, \\ |v| \leqslant (\log T)^2}} F(s+w) X^w \Gamma(w) dw \ll \frac{T^\epsilon T^{\delta/2 + \delta^2}}{X^{1/4}} = o(1) \tag{4.31}$$

since $\epsilon$ is arbitrarily small positive constant and $\delta$ is any small fixed positive constant. Therefore we get

$$F(s) = \sum_{T^\delta \leqslant n \leqslant T^{21\delta}} \frac{c_n}{n^s} e^{-\frac{n}{X}} + o(1). \tag{4.32}$$

Let

$$Z_1(\varrho) = \sum_{T^\delta \leqslant n \leqslant T^{21\delta}} \frac{c_n}{n^\varrho} e^{-\frac{n}{X}}. \tag{4.33}$$

Since $\varrho$ is a zero of $D(s), F(\varrho) = -1$ and hence,

$$|Z_1(\varrho) + o(1)| = |F(\varrho)| = 1 \Rightarrow |Z_1(\varrho)| > 1/2. \tag{4.34}$$

Let

$$Z_1(s) = \sum_{T^\delta \leqslant n \leqslant T^{21\delta}} \frac{b_n}{n^s}. \tag{4.35}$$

where

$$|b_n| = \left|c_n e^{-\frac{n}{X}}\right| \leqslant (d(n))^{30} n^{k-1}. \tag{4.36}$$

Let $U$ be a parameter with $T^\delta \leqslant U \leqslant T^{21\delta}$. Now, with $U = 2^j T^\delta$ for $j = 0, 1, 2, \cdots$, we have

$$|Z_1(\varrho)| = \sum_U \left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| > 1/2. \tag{4.37}$$

Note that $j \ll \log T$. Divide the width $[T, 2T]$ into abutting sub-intervals of width $(\log T)^2$ leaving a bit at the top. Call these smaller intervals as $I_1, I_2, \cdots$. Let

$$\mathcal{A} = \bigcup_{j=1,2,\cdots} I_{2j-1} \tag{4.38}$$

and

$$\mathcal{B} = \bigcup_{j=1,2,\cdots} I_{2j}. \tag{4.39}$$

Let $N_1$ and $N_2$ be the number of zeros in the sets $\mathcal{A}$ and $\mathcal{B}$ respectively. We first fix any smaller interval $I_{2j_0-1} \in \mathcal{A}$. The total number of zeros in this smaller interval $I_{2j_0-1}$ is $\ll (\log T)^3$. For every $\varrho \in I_{2j_0-1}$ and for every $\varrho' \in I_{2j-1}$ (with $j \neq j_0$), we have clearly

$$|\varrho - \varrho'| \geqslant (\log T)^2 .$$

Let

$$A^* = \{\varrho \in \mathcal{A} \ : \ |\varrho - \varrho'| \geqslant (\log T)^2 \text{ for } \varrho, \varrho' \in \mathcal{A}\}.$$

Let $N_1^*$ be the number of well spaced zeros $\varrho \in \mathcal{A}$. This implies that

$$N_1 \ll (\log T)^3 N_1^*.$$

Define

$$I(U) = \left\{ \varrho \in \mathcal{A}^* \ : \ \left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| \text{ is maximum} \right\}.$$

Now, $\mathcal{A}^*$ is the disjoint union of $I(U)$. (i.e) $\bigcup_U I(U) = \mathcal{A}^*$. Also we have a surjective map from the set $\{I(U)\}$ to the set $\{U\}$ with $I(U)$ is the inverse image of $U$. Similarly the same phenomena is true for $\mathcal{B}^*$.

Since for every $\varrho \in I(U)$, we have the sum $\left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right|$ is maximum, we obtain that for every $\varrho \in I(U)$,

$$\left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| > \frac{1}{2 \log T}. \tag{4.40}$$

Therefore

$$N^*\left(k - \frac{1}{4} + \delta^{1/2}, T, 2T\right) \ll (\log T)^3 \{N_1^* + N_2^*\}.$$

We notice that (for $U \leqslant n \leqslant 2U$)

$$e^{-\frac{n}{2U}} - e^{-\frac{n}{U}} = e^{-\frac{n}{U}} \left(e^{\frac{n}{2U}} - 1\right) > \frac{e^{1/2} - 1}{e^2} = C \tag{4.41}$$

and

$$\sum_{n=1}^{\infty} \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right) < \sum_{n=1}^{\infty} e^{-\frac{n}{2U}} = \frac{1}{e^{\frac{1}{2U}} - 1} < 2U. \tag{4.42}$$

We first treat the set $\mathcal{A}^*$. From (4.40) we have

$$\begin{aligned}\frac{N_1^*}{2\log T} &\leqslant \sum_U \sum_{\varrho\in I(U)} \eta_\varrho \sum_{U\leqslant n<2U} b_n n^{-\varrho}\\
&\leqslant \sum_{U\leqslant n<2U} |b_n| n^{-l} \left|\sum_{\varrho\in\mathcal{A}^*} \frac{\eta_\varrho}{n^{\varrho-l}}\right|\\
&\leqslant C^{-1}U^{-l+k-1+\epsilon} \sum_{U\leqslant n<2U} \left|\sum_{\varrho\in\mathcal{A}^*} \frac{\eta_\varrho}{n^{\varrho-l}}\right| \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right)\\
&\leqslant C^{-1}U^{-l+k-1+\epsilon} \left(\sum_{U\leqslant n<2U} \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right)\right)^{1/2}\\
&\quad\cdot \left(\sum_{U\leqslant n<2U} \sum_{\varrho_1,\varrho_2\in\mathcal{A}^*} \frac{\eta_{\varrho_1}\bar{\eta}_{\varrho_2}}{n^{\varrho_1+\bar{\varrho}_2-2l}} \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right)\right)^{1/2}\\
&\leqslant DU^{-l+k-1+\epsilon+\frac{1}{2}}L^{\frac{1}{2}} \qquad (4.43)\end{aligned}$$

where $\eta_{\varrho_1}$ and $\eta_{\varrho_2}$ are complex numbers of absolute value 1 whenever $\varrho_1, \varrho_2 \in \mathcal{A}^*$; 0 otherwise and

$$\begin{aligned}L &= \left(\sum_{U\leqslant n<2U} \sum_{\varrho_1,\varrho_2\in\mathcal{A}^*} \frac{\eta_{\varrho_1}\bar{\eta}_{\varrho_2}}{n^{\varrho_1+\bar{\varrho}_2-2l}} \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right)\right)\\
&= L_{\varrho_1=\varrho_2} + L_{\varrho_1\neq\varrho_2}. \qquad (4.44)\end{aligned}$$

$$\begin{aligned}|L_{\varrho_1=\varrho_2}| &< \sum_{U\leqslant n<2U} \sum_{\substack{\varrho_1=\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}^*}} \frac{1}{n^{\beta_1+\beta_2-2l}} \left(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}\right)\\
&\ll CN_1^* U^{1-2\delta^{1/2}} \qquad (4.45)\end{aligned}$$

since $\beta_1 + \beta_2 - 2l \geqslant 2\delta^{1/2}$. For $\varrho_1 \neq \varrho_2$, we observe that

$$|L_{\varrho_1\neq\varrho_2}| \leqslant \sum_{\substack{\varrho_1\neq\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}^*}} \left|\frac{1}{2\pi i} \int_{\Re w=1+\epsilon} \zeta(\varrho_1 + \bar{\varrho}_2 - 2l + w)\Gamma(w)(2^w - 1)U^w dw\right|. \qquad (4.46)$$

We notice that the contribution to the above integral from the portion $\Re w = 1 + \epsilon, |v| \geqslant (\log T)^2$ is $U^{1+\epsilon}e^{-C(\log T)^2}$ in absolute value. Therefore we get,

$$\begin{aligned}Q_1 =: &\sum_{\substack{\varrho_1\neq\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}^*}} \left|\frac{1}{2\pi i} \int_{\substack{\Re w=1+\epsilon,\\ |v|\geqslant(\log T)^2}} \zeta(\varrho_1 + \bar{\varrho}_2 - 2l + w)\Gamma(w)(2^w - 1)U^w dw\right|\\
&\ll N_1^{*2} U^{1+\epsilon} e^{-C(\log T)^2} \qquad (4.47)\end{aligned}$$

Note that the zeros are well-spaced. We move the line of integration to $\Re(\varrho_1+\bar{\varrho}_2-2l+w)=1/2$ in the remaining portion of the integral and notice that horizontal portions contribute a quantity which is $\ll N_1^{*2}T^{\delta^2}U^{1+\epsilon}e^{-C(\log T)^2}$ in absolute value, since by our assumption $\zeta(1/2+it)\ll t^{\delta^2}$. Now, on the vertical line, we find that

$$\begin{aligned}Q_2 =: \sum_{\substack{\varrho_1\neq\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}^*}}&\left|\frac{1}{2\pi i}\int\limits_{\substack{\Re w=1/2+2l-\Re(\varrho_1+\bar{\varrho}_2),\\ |v|\leqslant(\log T)^2}}\zeta(\varrho_1+\bar{\varrho}_2-2l+w)\Gamma(w)(2^w-1)U^w dw\right|\\ &\ll N_1^{*2}T^{\delta^2}U^{1/2+2l-\beta_1-\beta_2}\\ &\ll N_1^{*2}T^{\delta^2}U^{1/2-2\delta^{1/2}}\end{aligned}\tag{4.48}$$

since $\beta_1,\beta_2\geqslant l+\delta^{1/2}$. Therefore from (4.45), (4.47) and (4.48) we find that

$$\frac{N_1^*}{2\log T}\leqslant CU^{-l+k-\frac{1}{2}+\epsilon}\left\{N_1^*U^{1-2\delta^{1/2}}+N_1^{*2}T^{-20}+N_1^{*2}T^{\delta^2}U^{1/2-2\delta^{1/2}}\right\}^{1/2}.\tag{4.49}$$

We now fix $l=k-1/4$ and notice that

$$T^{\frac{\delta^2}{2}}U^{-l+k-1/2+\epsilon+1/4-\delta^{1/2}}N_1^*\ll\frac{T^{\frac{\delta^2}{2}}U^{\epsilon}}{U^{l-k+1/4+\delta^{1/2}}}N_1^*\ll\frac{N_1^*}{2(\log T)^2}$$

since $T^{\delta}\leqslant U\leqslant T^{21\delta}$, $\epsilon$ is any arbitrarily small positive constant and $\delta$ is any small but fixed positive constant. Also we notice that

$$N_1^{*2}T^{-10}\ll N_1^{*2}T^{\delta^2}U^{1/2-2\delta^{1/2}}.$$

Therefore we get

$$N_1^{*1/2}\leqslant C(\log T)U^{1/4-\delta^{1/2}+\epsilon}$$

and hence

$$N_1^*\ll T^{C\delta}$$

where the implied constant depends on $\delta,k,\epsilon$ but $C$ is independent of these parameters. Similar estimate holds for $N_2^*$. This proves the theorem.

**Proof of theorem 1.4.** Similar to the proof of the theorem 1.3

**Proof of theorem 1.5.** Since the proof of this theorem resembles the proof of the theorem 1.3 up to some extent, we only skectch the proof here. Let $\varrho=\beta+i\gamma, T\leqslant\gamma\leqslant 2T$ with $\beta\geqslant\sigma>k-1/4$. Let $X$ and $Y$ are two parameters which satisfy $X(\log X)^2\leqslant Y\leqslant T^A$. We define (as before)

$$F(s)=D(s)\sum_{n\leqslant X}\frac{\mu^*(n)}{n^s}-1=D(s)M_X(s)-1=\sum_{n>X}\frac{c_n}{n^s}\ \text{say}.\tag{4.50}$$

We notice that

$$c_n = 0 \quad \text{for} \quad n \leqslant X; |c_n| \leqslant (d(n))^{30} n^{k-1}. \tag{4.51}$$

Now, from Mellin's tranform (after truncating the integral at ($\Re w = k+1, |v| \geqslant (\log T)^2$) which gives an error $o(1)$) by moving the line of integration of the remaining portion to the line $\Re(s+w) = k - \frac{1}{2}$, we find that (for $\sigma > k - \frac{1}{4}, T \leqslant t \leqslant 2T$)

$$\begin{aligned} &\sum_{X \leqslant n \leqslant Y(\log Y)^2} \frac{c_n}{n^s} e^{-\frac{n}{Y}} + \sum_{n > Y(\log Y)^2} \frac{c_n}{n^s} e^{-\frac{n}{Y}} + o(1) \\ &= \frac{1}{2\pi i} \int\limits_{\substack{\Re w = k - \frac{1}{2} - \sigma, \\ |v| \leqslant (\log T)^2}} F(s+w) Y^w \Gamma(w) dw + F(s). \end{aligned} \tag{4.52}$$

Notice that (for $\sigma > k - 1/4$)

$$\sum_{n \geqslant Y(\log Y)^2} \frac{c_n}{n^s} e^{-\frac{n}{Y}} = o(1) \quad \text{as} \quad Y \to \infty \tag{4.53}$$

since $j > [Y(\log Y)^2]$. We note that

$$|M_X(k - \frac{1}{2} + ir)| \leqslant \sum_{n \leqslant X} \frac{(d(n))^{30} n^{k-1}}{n^{k-\frac{1}{2}}} \leqslant 2X^{\frac{1}{2}} (\log X)^{2^{30}}. \tag{4.54}$$

Therefore we have

$$\begin{aligned} Q_3 =: &\left| \frac{1}{2\pi i} \int\limits_{\substack{\Re w = k - \frac{1}{2} - \beta, \\ |v| \leqslant (\log T)^2}} F(\varrho + w) Y^w \Gamma(w) dw \right| \\ &< 2 \frac{\max\limits_{T - (\log T)^2 \leqslant r \leqslant T + (\log T)^2} |D(k - \frac{1}{2} + ir) M_X(k - \frac{1}{2} + ir)|}{Y^{\sigma - (k - \frac{1}{2})}} \\ &< 4 \frac{M_2(k - \frac{1}{2}, 2T) X^{\frac{1}{2}} (\log X)^{2^{30}}}{Y^{\sigma - (k - \frac{1}{2})}}. \end{aligned} \tag{4.55}$$

We choose $Y$ such that

$$4 \frac{M_2(k - \frac{1}{2}, 2T) X^{\frac{1}{2}} (\log X)^{2^{30}}}{Y^{\sigma - (k - \frac{1}{2})}} = \frac{1}{10}. \tag{4.56}$$

Let

$$Z_2(\varrho) = \sum_{X \leqslant n \leqslant Y(\log Y)^2} \frac{c_n}{n^{\varrho}} e^{-\frac{n}{Y}}. \tag{4.57}$$

Since $\varrho$ is a zero of $D(s), F(\varrho) = -1$. Therefore from (4.52), (4.53), (4.54), (4.56) and (4.57), we get

$$|Z_2(\varrho) + o(1)| > |F(\varrho)| - \frac{1}{10} \Rightarrow |Z_2(\varrho)| > 1/6. \tag{4.58}$$

Let

$$Z_2(s) = \sum_{X \leqslant n \leqslant Y(\log Y)^2} \frac{b_n}{n^s}. \tag{4.59}$$

where

$$|b_n| = \left|c_n e^{-\frac{n}{Y}}\right| \leqslant (d(n))^{30} n^{k-1}. \tag{4.60}$$

Let $U$ be a parameter with $X \leqslant U \leqslant Y(\log Y)^2$. With $U = 2^j X$ for $j = 1, 2, \cdots$ (note that $j \ll \log Y$), we have Now,

$$|Z_2(\varrho)| = \sum_U \left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| > 1/6. \tag{4.61}$$

As in the proof of the theorem 1.3, we obtain

$$\mathcal{A}_1 = \bigcup_{j=1,2,\cdots} I_{2j-1} \tag{4.62}$$

and

$$\mathcal{B}_1 = \bigcup_{j=1,2,\cdots} I_{2j}. \tag{4.63}$$

$$\mathcal{A}_1^* = \left\{\varrho \in \mathcal{A}_1 \ : \ |\varrho - \varrho'| \geqslant (\log T)^2 \text{ for } \varrho, \varrho' \in \mathcal{A}_1\right\}.$$

We define

$$I(U) = \left\{ \varrho \in \mathcal{A}_1{}^* \ : \ \left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| \text{ is maximum} \right\}.$$

Now, $\mathcal{A}_1{}^*$ is the disjoint union of $I(U)$. (i.e) $\bigcup_U I(U) = \mathcal{A}_1{}^*$. Also we have a surjective map from the set $\{I(U)\}$ to the set $\{U\}$ with $I(U)$ is the inverse image of $U$. Similarly the same phenomena is true for $\mathcal{B}_1{}^*$. Clearly

$$N^*(\sigma, T, 2T) \ll N_{11} + N_{12} \ll (\log T)^3 \{N_{11}^* + N_{12}^*\}, \tag{4.64}$$

where $N_{11}^*$ and $N_{12}^*$ be the number of zeros in the sets $\mathcal{A}_1^*$ and $\mathcal{B}_1^*$ respectively. Since for every $\varrho \in I(U)$, the sum $\left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right|$ is maximum, we obtain, for every $\varrho \in I(U)$,

$$\left| \sum_{U \leqslant n < 2U} \frac{b_n}{n^\varrho} \right| > \frac{1}{20 \log Y}. \tag{4.65}$$

We notice that (for $U \leqslant n \leqslant 2U$)

$$e^{-\frac{n}{2U}} - e^{-\frac{n}{U}} > C \ ; \ \sum_{n=1}^{\infty}(e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}) < 2U. \tag{4.66}$$

We first treat the set $\mathcal{A}_1^*$. From (4.65) we have

$$\begin{aligned}
\frac{N_{11}^*}{20\log Y} &\leqslant \sum_{\varrho\in U}\sum_{\varrho\in I(U)} \eta_\varrho \sum_{U\leqslant n<2U} b_n n^{-\varrho} \\
&\leqslant \sum_{U\leqslant n<2U} |b_n| n^{-l} \left| \sum_{\varrho\in\mathcal{A}_1^*} \frac{\eta_\varrho}{n^{\varrho-l}} \right| \\
&\leqslant \left( \sum_{U\leqslant n<2U} |b_n|^2 n^{-2l} \right)^{\frac{1}{2}} \left( \sum_{U\leqslant n<2U} \left| \sum_{\varrho\in\mathcal{A}_1^*} \frac{\eta_\varrho}{n^{\varrho-l}} \right|^2 \right)^{\frac{1}{2}} \\
&\leqslant C^{-\frac{1}{2}} \left( \sum_{U\leqslant n<2U} |b_n|^2 n^{-2l} \right)^{\frac{1}{2}} \left( \sum_{U\leqslant n<2U} \left| \sum_{\varrho\in\mathcal{A}_1^*} \frac{\eta_\varrho}{n^{\varrho-l}} \right|^2 (e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}) \right)^{\frac{1}{2}} \\
&\leqslant C^{-1/2} U^{-l+k-\frac{1}{2}} (\log U)^{2^{60}} \left( \sum_{U\leqslant n<2U} \left| \sum_{\varrho\in\mathcal{A}_1^*} \frac{\eta_\varrho}{n^{\varrho-l}} \right|^2 (e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}) \right)^{\frac{1}{2}} \\
&\leqslant C^{-1/2} U^{-l+k-\frac{1}{2}} (\log U)^{2^{60}} L^{\frac{1}{2}}
\end{aligned} \tag{4.67}$$

where $\eta_{\varrho_1}$ and $\eta_{\varrho_2}$ are complex numbers of absolute value 1 whenever $\varrho_1, \varrho_2 \in \mathcal{A}_1^*$; 0 otherwise and

$$\begin{aligned}
L &= \left( \sum_{U\leqslant n<2U} \sum_{\varrho_1,\varrho_2\in\mathcal{A}_1^*} \frac{\eta_{\varrho_1}\bar{\eta}_{\varrho_2}}{n^{\varrho_1+\bar{\varrho}_2-2l}} (e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}) \right) \\
&= L_{\varrho_1=\varrho_2} + L_{\varrho_1\neq\varrho_2}.
\end{aligned} \tag{4.68}$$

$$\begin{aligned}
|L_{\varrho_1=\varrho_2}| &< \sum_{U\leqslant n<2U} \sum_{\substack{\varrho_1=\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}_1^*}} \frac{1}{n^{\beta_1+\beta_2-2l}} (e^{-\frac{n}{2U}} - e^{-\frac{n}{U}}) \\
&\ll C N_{11}^* U
\end{aligned} \tag{4.69}$$

since $\beta_1 + \beta_2 - 2l \geqslant 0$. For $\varrho_1 \neq \varrho_2$, we observe that

$$|L_{\varrho_1\neq\varrho_2}| \leqslant \sum_{\substack{\varrho_1\neq\varrho_2,\\ \varrho_1,\varrho_2\in\mathcal{A}_1^*}} \left| \frac{1}{2\pi i} \int_{\Re w=1+\epsilon} \zeta(\varrho_1 + \bar{\varrho}_2 - 2l + w)\Gamma(w)(2^w - 1)U^w dw \right|. \tag{4.70}$$

We notice that the contribution to the above integral from the portion $\Re w = 1+\epsilon, |v| \geqslant (\log T)^2$ is $U^{1+\epsilon}e^{-C(\log T)^2}$ in absolute value. Therefore we get,

$$\sum_{\substack{\varrho_1 \neq \varrho_2, \\ \varrho_1, \varrho_2 \in \mathcal{A}_1^*}} \left| \frac{1}{2\pi i} \int_{\substack{\Re w = 1+\epsilon, \\ |v| \geqslant (\log T)^2}} \zeta(\varrho_1 + \bar{\varrho}_2 - 2l + w)\Gamma(w)(2^w - 1)U^w dw \right|$$
$$\ll {N_{11}^*}^2 U^{1+\epsilon} e^{-C(\log T)^2}. \tag{4.71}$$

Note that the zeros are well-spaced. We move the line of integration to $\Re(\varrho_1 + \bar{\varrho}_2 - 2l + w) = 1/2$ in the remaining portion of the integral and notice that horizontal portions contribute a quantity which is $\ll {N_{11}^*}^2 T^{1/6} U^{1+\epsilon} e^{-C(\log T)^2}$ in absolute value, since $\zeta(1/2 + it) \ll t^{1/6}$. Now, on the vertical line, we find that

$$\sum_{\substack{\varrho_1 \neq \varrho_2, \\ \varrho_1, \varrho_2 \in \mathcal{A}_1^*}} \left| \frac{1}{2\pi i} \int_{\substack{\Re w = 1/2 + 2l - \Re(\varrho_1 + \bar{\varrho}_2), \\ |v| \leqslant (\log T)^2}} \zeta(\varrho_1 + \bar{\varrho}_2 - 2l + w)\Gamma(w)(2^w - 1)U^w dw \right|$$
$$\ll {N_{11}^*}^2 M_1(\frac{1}{2}, 2T) U^{1/2 + 2l - \beta_1 - \beta_2}$$
$$\ll {N_{11}^*}^2 M_1(\frac{1}{2}, 2T) U^{1/2} \tag{4.72}$$

since $\beta_1, \beta_2 \geqslant l$. Therefore from (4.69), (4.71) and (4.72) we find that

$$\frac{N_{11}^*}{20 \log Y} \leqslant C^{-1/2} U^{-l+k-\frac{1}{2}} (\log U)^{2^{60}}$$
$$\times \left\{ N_{11}^* U + {N_{11}^*}^2 T^{-20} + {N_{11}^*}^2 M_1(\frac{1}{2}, 2T) U^{1/2} \right\}^{1/2}. \tag{4.73}$$

We take $l = \sigma$ and notice that

$${N_{11}^*}^2 T^{-10} \leqslant {N_{11}^*}^2 M_1(\frac{1}{2}, 2T) U^{1/2}.$$

Now we choose $X$ such that

$$2C^{-1/2} \left( M_1(\frac{1}{2}, 2T) \right)^{1/2} X^{-\sigma + k - \frac{1}{4}} (\log X)^{2^{60}} = \frac{1}{20 \log Y}. \tag{4.74}$$

This implies that

$$\frac{N_{11}^*}{20 A \log T} \leqslant C^{-1/2} U^{k-\sigma} {N_{11}^*}^{1/2}$$

and hence we obtain

$$N_{11}^{*} \ll U^{2(k-\sigma)}(\log T)^2 \ll Y^{2(k-\sigma)}(\log T)^2. \tag{4.75}$$

From our choice of $X$ and $Y$ (see (4.74) and (4.56)), we find that

$$N_{11}^{*} \ll \left(M_2^2(M_1^{\frac{1}{2}}(\log T)^{2^{61}})^{\frac{4}{4\sigma-4k+1}}(\log T)^{2^{31}}\right)^{\frac{2(k-\sigma)}{2\sigma-2k+1}}(\log T)^2. \tag{4.76}$$

Similar estimate holds for $N_{12}^{*}$. Hence the theorem follows.

**Remark.** It is easy to see that theorem 1.5 is better than theorem 1.1 whenever $\sigma$ is nearer to $k$. If $N_Z(\sigma,T)$ denotes the number of zeros $\varrho = \beta + i\gamma$ of the Rankin-Selberg zeta function $Z(s)$ with $\beta \geqslant \sigma$ and $|\gamma| \leqslant T$, then (after normalizing the result of theorem 1.5), from lemma 3.1 (we have $Z(s) = \zeta(s)D(s+k-1)$) and theorem A, we find that (for $\sigma > \frac{3}{4}$)

$$N_Z(\sigma,T) \ll \left(M_2^2(M_1^{\frac{1}{2}}(\log T)^{2^{61}})^{\frac{4}{4\sigma-3}}(\log T)^{2^{31}}\right)^{\frac{2(1-\sigma)}{2\sigma-1}}(\log T)^5 \tag{4.77}$$

whenever $M_2 > M_1^{\frac{6\sigma-5}{4\sigma-3}}$ and

$$N_Z(\sigma,T) \ll \left(M_1(\log T)^6\right)^{\frac{8(1-\sigma)(3\sigma-2)}{(4\sigma-3)(2\sigma-1)}}(\log T)^{11} \tag{4.78}$$

whenever $M_2 < M_1^{\frac{6\sigma-5}{4\sigma-3}}$. However it should be mentioned here again that all we know only about $M_1$ and $M_2$ are $M_1 \ll T^{\frac{89}{570}+\epsilon}$ and $M_2 \ll T^{\frac{3}{4}}\log T$.

## 5. Concluding Remarks

Most of the theorems in this paper automatically hold in the case of Rankin-Selberg zeta-functions because of the relation from lemma 3.1. It would be much more interesting to avoid the condition $\zeta(\frac{1}{2}+it) \ll t^{\delta^2}$ in the theorems 1.3 and 1.4, but we do not know how to do it. The general theorem 1.4 may be compared with the general theorem 2 of [13]. To prove theorem 2 of [13], Turan's power sum method (precisely the second main theorem of Turan's Power sum method) plays an important role. However we have avoided this in our proof with slightly modifying the assumptions on $G(s)$.

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