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CONVERGENCE IN BV_φ BY NONLINEAR MELLIN-TYPE CONVOLUTION OPERATORS

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Abstract: In this paper we establish convergence results for a family $\mathbb{T}$ of nonlinear integral operators of the form:

$$(T_w f)(s) = \int_0^{+\infty} K_w(t, f(st))dt = \int_0^{+\infty} L_w(t) H_w(f(st))dt, \quad s \in \mathbb{R}_0^+,$$

where $f \in Dom\mathbb{T}$, $Dom\mathbb{T}$ being the class of all the measurable functions $f: \mathbb{R}_0^+ \to \mathbb{R}$ such that $T_w f$ is well defined as Lebesgue integral for every $s \in \mathbb{R}_0^+$. For the above family of nonlinear Mellin type operators, under suitable singularity assumptions on the kernels $\mathbb{K} = \{K_w\}$, we state a convergence result of type $\lim_{w \to +\infty} V_\varphi[\mu(T_w f - f)] = 0$, for some constant $\mu > 0$ and for every f belonging to a suitable subspace of BV_φ-functions.
Keywords: Musielak-Orlicz φ-variation, V_φ-convergence, locally φ, η-absolutely continuous functions, nonlinear Mellin type convolution operators.

1. Introduction

In [16] there is considered convergence with respect to φ-variation and rate of approximation for a class of linear integral operators of the form:

$$(T_w f)(s) = \int_{\mathbb{R}^+} K_w(s,t)\ f(t)\ dt, \tag{I}$$

defined for every $f \in X$ for which $(T_w f)(s)$ is well-defined for every $s \in \mathbb{R}^+$ and for every $w > 0$, being $K_w : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}_0^+$ a family of kernel functions satisfying a general homogeneity condition with respect to a measurable function η, and where X denotes the space of all Lebesgue measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}$. Results concerning estimates for operators of the form (I) with respect to φ-variation in one-dimensional and in multidimensional frame can be found in [3], [4], [17].

The concept of φ-variation, has been introduced by L.C. Young in [18] and in [14] this concept was developed by J. Musielak and W. Orlicz in the direction of function spaces; it represents a generalization of the classical Jordan variation.

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Given a φ-function $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, for every $f \in X$, the *Musielak-Orlicz φ-variation* of f is defined as

$$V_\varphi[f] = V_\varphi[f; \mathbb{R}^+] = \sup_\Pi \sum_{i=1}^{n} \varphi(|f(t_i) - f(t_{i-1})|)$$

where the supremum is taken over all finite increasing sequences Π in $\mathbb{R}_0^+$ (see [14], [12] in case of a bounded interval). By means of this functional it is possible to define the space of functions with bounded φ-variation on $\mathbb{R}_0^+$ in the sense of Musielak-Orlicz, as

$$BV_\varphi(\mathbb{R}_0^+) = \{f \in X : \lim_{\lambda \to 0} V_\varphi[\lambda f] = 0\}.$$

It is possible to observe that the functional $\rho : X \to [0, +\infty]$, defined by

$$\rho(f) = V_\varphi[f] + |f(a)|,$$

for some $a \geq 0$, $f \in X$, is a convex modular on X; therefore the space $BV_\varphi(\mathbb{R}_0^+)$ is connected with the theory of modular space and hence also the formulation of convergence in φ-variation is connected with the modular convergence (see [15], [12], [10]). Namely we will say that

a family $(f_w)_{w \in \mathbb{R}^+} \in BV_\varphi(\mathbb{R}_0^+)$ is said to be convergent in φ-variation to $f \in BV_\varphi(\mathbb{R}_0^+)$ if there exists a $\lambda > 0$ such that $V_\varphi[\lambda(f_w - f)] \to 0$ as $w \to +\infty$.

The problem of convergence in φ-variation for a family of nonlinear integral operators is very delicate. Indeed, the modular ρ above introduced, does not satisfy the assumptions which are generally used in modular convergence problems of various families of this kind of operators (see e.g. [13], [1], [5]). In this paper, using a different approach, we will study properties of convergence in $BV_\varphi(\mathbb{R}_0^+)$ for the family of nonlinear integral operators of Mellin-type:

$$(T_w f)(s) = \int_0^{+\infty} K_w(t, f(st))dt = \int_0^{+\infty} L_w(t) H_w(f(st))dt \quad s \in \mathbb{R}_0^+, \qquad (II)$$

where $f \in Dom\mathbb{T}$, being $Dom\mathbb{T}$ the class of all measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}$ such that $T_w f$ is well defined as Lebesgue integral for every $s \in \mathbb{R}_0^+$. The above operators represent a nonlinear version of *linear* convolution Mellin-type operators, which are considered in the classical theory of Mellin Transform (see [6], [7]).

For estimates with respect to φ-variation (also in the generalized sense) for operators of type (II), see [11].

The main result of the paper is a convergence theorem (Theorem 2) which states that for $f \in AC_{loc}^{\varphi}(\mathbb{R}^+) \cap BV_{\varphi+\eta}(\mathbb{R}_0^+)$, and under singularity assumptions on the kernels $\mathbb{K} = \{K_w\}$, there is a constant $\mu > 0$ sufficiently small that

$$\lim_{w \to +\infty} V_\varphi[\mu(T_w f - f)] = 0,$$

that is the family of nonlinear integral operators converges with respect to φ-variation towards f. Here φ and η are two φ-functions satisfying suitable assumptions. In order to formulate the convergence theorem (Theorem 2) there are of fundamental importance the convergence in φ-variation for the dilation operator τ_z calculated over $(H_w \circ f)$, as $z \to 1$ (Theorem 1) and an equiboundedness in φ- variation for the family $\{H_w \circ f\}$ (Lemma 3) together with the result (Lemma 3) that for every $\varepsilon > 0$ there exists a step function $\nu : \mathbb{R}_0^+ \to \mathbb{R}$ such that

$$V_\varphi[\lambda(H_w \circ f - \nu), [0, b]] < \varepsilon$$

for a suitable $\lambda > 0$, uniformly with respect $w \geq \overline{w} > 0$ and for every interval $[0, b]$, and being $f \in AC_{loc}^\varphi(\mathbb{R}_0^+) \cap BV_\varphi(\mathbb{R}_0^+)$.

2. Preliminaries

Let X be the space of all Lebesgue measurable functions $f : \mathbb{R}_0^+ \to \mathbb{R}$ where $\mathbb{R}_0^+ = [0, +\infty)$.
Let Φ be the class of all nondecreasing functions $\varphi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following assumptions:

i) $\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$;
ii) φ is a convex function on $\mathbb{R}_0^+$;
iii) $u^{-1}\varphi(u) \to 0$ as $u \to 0^+$.

From now on we will always suppose that $\varphi \in \Phi$ and we will say that φ is a φ-function.
Now, for every $f \in X$, we define the *Musielak-Orlicz φ-variation* of f as follows

$$V_\varphi[f] = V_\varphi[f; \mathbb{R}^+] = \sup_{\Pi} \sum_{i=1}^{n} \varphi(|f(t_i) - f(t_{i-1})|)$$

where Π denotes an increasing finite sequence in $\mathbb{R}_0^+$ (see [14], [12]).
It is easy to see that the functional $\rho : X \to [0, +\infty]$, defined by

$$\rho(f) = V_\varphi[f] + |f(a)|,$$

for some $a \geq 0$, $f \in X$, is a convex *modular* on X (see [12]).
In the following we will identify functions which differ from a constant.
By means of the above modular ρ, we define the space of functions with bounded φ-variation on $\mathbb{R}_0^+$ in the sense of Musielak-Orlicz, as

$$BV_\varphi(\mathbb{R}_0^+) = \{f \in X : \lim_{\lambda \to 0} \rho(\lambda f) = 0\} = \{f \in X : \lim_{\lambda \to 0} V_\varphi[\lambda f] = 0\}.$$

It is possible to observe that by monotonicity and convexity of φ, we have

$$BV_\varphi(\mathbb{R}_0^+) = \{f \in X : \exists \lambda > 0 : V_\varphi[\lambda f] < +\infty\},$$

and there results that if $f \in BV_\varphi(\mathbb{R}_0^+)$, then f is bounded in $\mathbb{R}_0^+$. In the following we will denote $BV_\varphi(\mathbb{R}_0^+)$ simply by BV_φ.

We will say that a family of functions $\{f_w\}_{w>0}$ is of *equibounded φ-variation* if it is of bounded φ-variation uniformly with respect to $w > 0$.

Now we recall the following result about φ-variation, which we will use in the following (see [14], [2]):

j) if $f_1, f_2, \ldots, f_n \in X$, then

$$V_\varphi[\sum_{i=1}^{n} f_i] \leq \frac{1}{n} \sum_{i=1}^{n} V_\varphi[n f_i].$$

Let $\varphi, \eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be two $\varphi-functions$. We will say that a function $f : \mathbb{R}_0^+ \to \mathbb{R}$ is *locally (φ,η)-absolutely continuous* if there is a $\lambda > 0$ such that the following property holds: for every $\varepsilon > 0$ and every bounded interval $J \subset \mathbb{R}_0^+$, there is a $\delta > 0$ such that for any finite collection of non-overlapping intervals $[a_i, b_i] \subset J$, $i = 1, 2, \ldots, N$, with $\sum_{i=1}^{N} \varphi(b_i - a_i) < \delta$ there results

$$\sum_{i=1}^{N} \eta(\lambda|f(b_i) - f(a_i)|) < \varepsilon. \tag{1}$$

If $\eta = \varphi$ in the above property, we will say that f is locally φ- absolutely continuous (see [14], [12], [16]), and we will denote by $AC_{loc}^{\varphi}(\mathbb{R}_0^+)$ the class of all these functions.

We will say that a family of functions $\{f_w\}_{w>0}$ is *locally equi (φ,η)-absolutely continuous* if there is $\lambda > 0$ such that for every $\varepsilon > 0$ and every bounded interval $J \subset \mathbb{R}_0^+$, we can choose a $\delta > 0$ for which the local absolute φ-continuity of f_w holds uniformly with respect to $w > 0$. For $\eta = \varphi$ we will speak of local equi φ-absolute continuity.

Let now $\mathcal{K}$ be the class of all the functions $K : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$ of the form

$$K(t,u) = L(t)H(u), \quad t \in \mathbb{R}_0^+, \ u \in \mathbb{R},$$

where $L \in L^1(\mathbb{R}_0^+)$, $L \geq 0$ and $H : \mathbb{R} \to \mathbb{R}$ is a function satisfying a Lipschitz condition of type

$$|H(u) - H(v)| \leq \psi(|u - v|), \quad u, v \in \mathbb{R}, \tag{2}$$

where $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function with the following properties:

1. $\psi(0) = 0$, $\psi(u) > 0$ for $u > 0$;
2. ψ is continuous and nondecreasing.

We will denote with Ψ the class of all functions ψ satisfying the above conditions.

Let $\mathbb{K} = \{K_w\}_{w>0}$ be a set of functions from $\mathcal{K}$, $K_w(t,u) = L_w(t)H_w(u)$, $w > 0$, $t \in \mathbb{R}_0^+$, $u \in \mathbb{R}$. We will say that $\mathbb{K}$ is *singular* in $BV_\varphi(\mathbb{R}_0^+)$, if the following assumptions hold:

(K.1) there exists $A > 0$, such that $0 < \|L_w\|_1 = A_w \le A$ for every $w > 0$;

(K.2) for every $\delta \in (0,1)$, we have

$$\lim_{w \to +\infty} \int_{|1-t|>\delta} L_w(t)dt = 0;$$

(K.3) putting $G_w(u) = H_w(u) - u$, for every $u \in \mathbb{R}$, $w > 0$, there exists $\lambda > 0$ such that

$$V_\varphi[\lambda G_w, J] \to 0, \text{ as } w \to +\infty,$$

for every bounded interval $J \subset \mathbb{R}_0^+$.

Example 1. For every $n \in \mathbb{N}$, let

$$K_n(t,u) = L_n(t)H_n(u), \quad t \in \mathbb{R}_0^+, \ u \in \mathbb{R},$$

where

$$H_n(u) = \begin{cases} n\log(1+u/n), & 0 \le u < 1 \\ nu\log(1+1/n), & u \ge 1, \end{cases}$$

where we extend in odd-way the definition of H_n for $u < 0$; moreover $\{L_n\}_{n\in\mathbb{N}}$ is a classical kernel with the mass concentrated at 1, i.e.

$$\int_0^\infty L_n(t)dt = 1, \text{ for every } n \in \mathbb{N},$$

with the property (K.2). It is easy to show that

$$|H_n(u) - H_n(v)| \le |u - v|, \text{ for every } u, v \in \mathbb{R}, \text{ and } n \in \mathbb{N}$$

and, for every $u \ge 0$, we have

$$|G_n(u)| = |H_n(u) - u| = \begin{cases} u - n\log(1+u/n), & 0 \le u < 1 \\ u[1 - n\log(1+1/n)], & u \ge 1. \end{cases}$$

Then $|G_n(u)|$ is increasing on $\mathbb{R}_0^+$. If φ: $\mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a convex function, using Proposition 1.03 in [14], we have, for every interval $J = [0, M]$,

$$V_\varphi[G_n, J] = \varphi(|G_n(M) - G_n(0)|) \to 0, \text{ as } n \to +\infty.$$

Analogously, by the definition of H_n for $u < 0$, we have $V_\varphi[G_n, [-M, 0]] \to 0$, as $n \to +\infty$.

3. Preliminary lemmas

Before to formulate the following lemmas, we recall the concept of convergence in φ-variation (see [14], [12], [2], [16]).

We say that a sequence $(f_w)_{w\in\mathbb{R}^+} \in BV_\varphi$ is *convergent in φ-variation to* $f \in BV_\varphi$ if there exists a $\lambda > 0$ such that $V_\varphi[\lambda(f_w - f)] \to 0$ as $w \to +\infty$. Moreover we will use the following relation between the functions φ, ψ and η, being φ, η two φ-functions, with η not necessarily convex and $\psi \in \Psi$.

We say that the triple $\{\varphi, \eta, \psi\}$ is *properly directed*, if the following condition holds (for similar assumptions see [11]): for every $\lambda > 0$, there exists a constant C_λ such that

$$\varphi(C_\lambda \psi(u)) \leq \eta(\lambda u), \text{ for every } u \geq 0. \tag{3}$$

Now we start to formulate the following lemma.

Lemma 1. *Let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a locally (φ, η)-absolutely continuous function. Let $\{H_w\}_{w>0}$ be a class of functions satisfying (2) for a fixed $\psi \in \Psi$ and let us assume that the triple $\{\varphi, \eta, \psi\}$ is properly directed.*
Then the family $\{H_w \circ f\}_{w>0}$ is locally equi φ-absolutely continuous.

Proof. Let $\lambda > 0$ be a constant for which the definition of the (φ, η)-absolute continuity of f holds and let $0 < \mu \leq C_\lambda$, being C_λ the constant in (3). Since f is locally (φ, η)-absolutely continuous, for a fixed interval $J \subset \mathbb{R}_0^+$ and $\varepsilon > 0$ there is a $\delta > 0$ such that (1) holds for any finite collection of intervals $I_i = [a_i, b_i]$, $i = 1, 2, \dots N$, with $\sum_{i=1}^N \varphi(b_i - a_i) < \delta$. For such a family $\{I_i\}$, we have

$$\begin{aligned} &\sum_{i=1}^N \varphi(\mu|(H_w \circ f)(b_i) - (H_w \circ f)(a_i)|) \\ &\leq \sum_{i=1}^N \varphi(C_\lambda \psi(|f(b_i) - f(a_i)|)) \\ &\leq \sum_{i=1}^N \eta(\lambda|f(b_i) - f(a_i)|) < \varepsilon. \end{aligned}$$

■

Lemma 2. *Let f be a locally φ-absolutely continuous function such that $f \in BV_\varphi(\mathbb{R}_0^+)$. Let $\{H_w\}_{w>0}$ be a family of functions $H_w : \mathbb{R} \to \mathbb{R}$ such that (K.3) holds. Then there is $\lambda > 0$ such that the following property holds: for every $\varepsilon > 0$ and every interval $[0, b] \subset \mathbb{R}_0^+$, there are a $\overline{w} > 0$ and a step function $\nu : \mathbb{R}_0^+ \to \mathbb{R}$ such that*

$$V_\varphi[\lambda(H_w \circ f - \nu), [0, b]] < \varepsilon$$

uniformly with respect $w \geq \overline{w} > 0$.

Proof. Let $[0, b] \subset \mathbb{R}_0^+$ be a fixed bounded interval. From Lemma 1 in [16], (see also Theorem 2.21 of [14]), there is a $\lambda > 0$ such that, for a fixed $\varepsilon > 0$ there exists a division $D = \{\tau_0 = 0, \tau_1, \dots, \tau_n = b\}$ of the interval $[0,b]$, such that the step function $\nu : \mathbb{R}_0^+ \to \mathbb{R}$, defined by

$$\nu(t) = \begin{cases} f(\tau_{i-1}), & \tau_{i-1} \le t < \tau_i, \qquad i = 1, \dots m \\ f(b), & t \ge b \end{cases}$$

satisfies

$$V_\varphi[2\lambda(f-\nu), [0,b]] < \varepsilon/2.$$

Now, let $D = \{t_0, t_1, \dots, t_n\}$ be an arbitrary partition of $[0,b]$, with $t_0 < t_1 < \dots < t_n$. We have

$$\begin{aligned} &\sum_{i=1}^{n} \varphi(\lambda |H_w(f(t_i)) - \nu(t_i) - \{H_w(f(t_{i-1})) - \nu(t_{i-1})\}|) \\ &\le \frac{1}{2} \sum_{i=1}^{n} \varphi(2\lambda |H_w(f(t_i)) - f(t_i) - \{H_w(f(t_{i-1})) - f(t_{i-1})\}|) \\ &+ \frac{1}{2} \sum_{i=1}^{n} \varphi(2\lambda |f(t_i) - \nu(t_i) - \{f(t_{i-1}) - \nu(t_{i-1})\}|) \\ &= I_1 + I_2. \end{aligned}$$

Since $f \in BV_\varphi(\mathbb{R}_0^+)$, f is bounded, i.e. there is $M > 0$ such that $|f(t)| \le M$. Putting $J = [-M, M]$, we have

$$I_1 \le \frac{1}{2} V_\varphi[2\lambda G_w, J].$$

Thus using (K.3) we can take $\lambda > 0$ such that $I_1 \le \varepsilon/2$ for sufficiently large $w > 0$. The assertion follows being $I_2 \le \frac{1}{2} V_\varphi[2\lambda(f-\nu), [0,b]] < \varepsilon/2$. ■

Lemma 3. *Let $f \in BV_\eta(\mathbb{R}_0^+)$ and $\{H_w\}$ be a family of functions $H_w : \mathbb{R} \to \mathbb{R}$ satisfying (2). Let us suppose that the triple $\{\varphi, \eta, \psi\}$ is properly directed. Then the family $\{H_w \circ f\}$ is of equibounded φ-variation on every interval $I^* \subset \mathbb{R}_0^+$.*

Proof. Let $D = \{t_0, t_1, \dots t_n\} \subset I^*$ be fixed and let $\lambda > 0$. For $0 < \mu \le C_\lambda$, C_λ being the constant in (3), we have

$$\begin{aligned} &\sum_{i=1}^{n} \varphi(\mu |(H_w \circ f)(t_i) - (H_w \circ f)(t_{i-1})|) \\ &\le \sum_{i=1}^{n} \varphi(C_\lambda \psi(|f(t_i) - f(t_{i-1})|). \end{aligned}$$

Now, by (3) we have

$$\begin{aligned} &\sum_{i=1}^{n} \varphi(\mu |(H_w \circ f)(t_i) - (H_w \circ f)(t_{i-1})|) \\ &\le \sum_{i=1}^{n} \eta(\lambda |f(t_i) - f(t_{i-1})|) \le V_\eta[\lambda f, I^*], \end{aligned}$$

and so the assertion follows. ■

4. An approximation result by means of the dilation operator

For any $z \in \mathbb{R}^+$, we will put:

$$\tau_z f(s) = f(sz),$$

for every $f : \mathbb{R}_0^+ \to \mathbb{R}$ and $s \in \mathbb{R}_0^+$. Using the above lemmas, we show the following theorem

Theorem 1. *Let φ, η be fixed and let $f : \mathbb{R}_0^+ \to \mathbb{R}$ be a locally φ-absolutely continuous function, such that $f \in BV_{\varphi+\eta}(\mathbb{R}_0^+)$. Let $\{H_w\}$ be a family of functions $H_w : \mathbb{R} \to \mathbb{R}$ satisfying (K.3) and (2) for a fixed $\psi \in \Psi$. Let us assume that the triple $\{\varphi, \eta, \psi\}$ is properly directed.Then for every $\lambda > 0$ there exist a constant $\mu > 0$ and $\overline{w} > 0$ such that*

$$\lim_{z \to 1} V_\varphi[\mu(\tau_z(H_w \circ f) - (H_w \circ f))] = 0$$

uniformly with respect to $w \geq \overline{w}$.

Proof. Let $g_w = H_w \circ f$, for $w > 0$. Since $f \in BV_\eta(\mathbb{R}_0^+)$, from Lemma 1 of [16], given $\varepsilon > 0$ there is $\mathrm{c} > 0$ and $\lambda_0 > 0$ such that $V_\eta[\lambda f, [\mathrm{c}, +\infty)] < \varepsilon$, for every $0 < \lambda \leq \lambda_0$. From Lemma 3, there exists a constant $\mu > 0$ so small that

$$V_\varphi[4\mu g_w, [\mathrm{c}, +\infty)] \leq V_\eta[\lambda f, [\mathrm{c}, +\infty)] < \varepsilon$$

uniformly with respect to $w > 0$. Let us choose constants d, b with $d > b > \mathrm{c}$ and let ν be a step function on $[0, d]$ given in Lemma 2. Let now z be such that $\mathrm{c}/b < z < \min\{d/b, b/\mathrm{c}\}$. By convexity of φ, and property j), for every z sufficiently near to 1, we have now, for sufficiently small $\mu > 0$,

$$\begin{aligned}
&V_\varphi[\mu(\tau_z g_w - g_w)] \\
&\leq \frac{1}{2}\{V_\varphi[2\mu(\tau_z g_w - g_w), [0, b]] + V_\varphi[2\mu(\tau_z g_w - g_w), [b, +\infty)]\} \\
&\leq \frac{1}{2}V_\varphi[2\mu(\tau_z g_w - g_w), [0, b]] + \frac{1}{4}\{V_\varphi[4\mu(\tau_z g_w), [b, +\infty)] + V_\varphi[4\mu(g_w), [b, +\infty)]\} \\
&\leq \frac{1}{2}V_\varphi[2\mu(\tau_z g_w - g_w), [0, b]] + \frac{1}{2}\, V_\eta[\lambda f, [\mathrm{c}, +\infty)] \\
&\leq \frac{1}{2}V_\varphi[2\mu(\tau_z g_w - g_w), [0, b]] + \varepsilon.
\end{aligned}$$

The first inequality comes from a classical property of φ-variation (see [14], Proposition 1.17).
Now we consider the interval $I^* = [0, b]$. We have, for sufficiently small $\mu > 0$,

$$\begin{aligned}
&V_\varphi[2\mu(\tau_z g_w - g_w), I^*] \\
&\leq \frac{1}{3}\{V_\varphi[6\mu\tau_z(g_w - \nu), I^*] + V_\varphi[6\mu(\nu - g_w), I^*] + V_\varphi[6\mu(\tau_z\nu - \nu), I^*]\} \\
&\leq \frac{1}{3}\{2V_\varphi[6\mu(g_w - \nu), [0, d]] + V_\varphi[6\mu(\tau_z\nu - \nu), [0, d]]\} \\
&= I_1 + I_2.
\end{aligned}$$

Now from Lemma 2, $I_1 \le \varepsilon/2$, while as in Theorem 1 in [2], we have $I_2 \le \varepsilon/2$. Thus the assertion follows. ■

5. An approximation theorem for nonlinear Mellin-type convolution operators

Let $\mathbb{K} = \{K_w(t,u)\}_{w>0}$ be a singular kernel in $BV_\varphi(\mathbb{R}_0^+)$, where, as before, $K_w(t,u) = L_w(t)H_w(u)$ for $t \in \mathbb{R}_0^+, u \in \mathbb{R}$ and $w > 0$.

We will study approximation properties of the family of nonlinear integral operators $\mathbb{T} = \{T_w\}$ defined by

$$(T_w f)(s) = \int_0^{+\infty} K_w(t, f(st))dt = \int_0^{+\infty} L_w(t)H_w(f(st))dt \quad s \in \mathbb{R}_0^+,$$

where $f \in Dom\mathbb{T}$. Let us remark here that if the function f is such that $(H_w \circ f) \in L^1(\mathbb{R}_0^+)$, or if $f \in L^\infty(\mathbb{R}_0^+)$, then $f \in Dom\mathbb{T}$. So in particular, if f is of bounded φ-variation, where φ is an arbitrary φ-function, $f \in Dom\mathbb{T}$.

Let now φ, η be two φ-functions, with η not necessarily convex, such that the triple $\{\varphi, \eta, \psi\}$ is properly directed. Then in [11] it is proved that if $f \in BV_\eta(\mathbb{R}_0^+)$ then $T_w f$ is of bounded φ-variation, for every $w > 0$.

We have the following

Theorem 2. *Let $f \in AC_{loc}^\varphi(\mathbb{R}_0^+) \cap BV_{\varphi+\eta}(\mathbb{R}_0^+)$ and let us assume that the triple $\{\varphi, \eta, \psi\}$ is properly directed. Let $\mathbb{K} = \{K_w\} \subset \mathcal{K}$ be a singular kernel in $BV_\varphi(\mathbb{R}_0^+)$. Then there exists a constant $\mu > 0$ such that*

$$\lim_{w \to +\infty} V_\varphi[\mu(T_w f - f)] = 0.$$

Proof. First of all we remark that $T_w f - f \in BV_\varphi(\mathbb{R}_0^+)$. We can assume that $A_w = 1$, for every $w > 0$, where A_w are the constants given in (K.1). Let $\lambda > 0$ such that $V_\eta[\lambda f] < +\infty$, and let $\mu > 0$ so small that $4\mu \le C_\lambda$ and

$$\lim_{z \to 1} V_\varphi[2\mu(\tau_z(H_w \circ f) - (H_w \circ f))] = 0,$$

uniformly with respect to sufficiently large $w > 0$ (Theorem 1).

Let $D = \{s_0, s_1, \ldots, s_N\} \subset \mathbb{R}_0^+$ be a finite increasing sequence and let μ sufficiently small. We have:

$$\sum_{i=1}^N \varphi[\mu|(T_w f)(s_i) - (T_w f)(s_{i-1}) - f(s_i) + f(s_{i-1})|]$$
$$= \sum_{i=1}^N \varphi[\mu| \int_0^{+\infty} L_w(t)[H_w(f(s_i t)) - H_w(f(s_i))$$
$$+ H_w(f(s_i)) - f(s_i) - H_w(f(s_{i-1}t)) + H_w(f(s_{i-1})) - H_w(f(s_{i-1}) + f(s_{i-1})]dt|]$$

$$\leq \frac{1}{2} \sum_{i=1}^{N} \int_0^{+\infty} L_w(t)\varphi[2\mu|(H_w(f(s_i t))$$
$$- H_w(f(s_i))) - (H_w(f(s_{i-1}t)) - H_w(f(s_{i-1})))|]dt$$
$$+ \frac{1}{2} \sum_{i=1}^{N} \int_0^{+\infty} L_w(t)\varphi[2\mu|(H_w(f(s_i)) - f(s_i)) - (H_w(f(s_{i-1})) - f(s_{i-1}))|]dt$$
$$= I_1 + I_2.$$

Now given $\delta \in (0,1)$, we write

$$I_1 \leq \frac{1}{2} \sum_{i=1}^{N} \left\{ \int_{|1-t|<\delta} + \int_{|1-t|>\delta} \right\}$$
$$L_w(t)\varphi[2\mu|(H_w(f(s_i t)) - H_w(f(s_i))) - (H_w(f(s_{i-1}t)) - H_w(f(s_{i-1})))|]dt$$
$$= I_1^1 + I_1^2.$$

Next,

$$I_1^1 \leq \frac{1}{2} \int_{1-\delta}^{1+\delta} L_w(t)V_\varphi[2\mu[\tau_t(H_w \circ f) - (H_w \circ f)]]dt$$

and so, for sufficiently small $\delta \in (0,1)$ we have $I_1^1 \leq \varepsilon$, uniformly with respect to $w > 0$.

Now, by property j),

$$I_1^2 \leq \frac{1}{4} \int_{|1-t|>\delta} L_w(t)V_\varphi[4\mu(H_w \circ f)]dt$$
$$\leq \frac{1}{4} V_\eta[\lambda f] \int_{|1-t|>\delta} L_w(t)dt,$$

and so, from (K.2), $I_1^2 \to 0$, as $w \to +\infty$.

Finally, we estimate I_2. We have:

$$I_2 \leq \frac{1}{2} \int_0^{+\infty} L_w(t)V_\varphi[2\mu G_w] = \frac{1}{2} V_\varphi[2\mu G_w].$$

But since f is bounded, there is $M > 0$, such that $|f(t)| \leq M$ for every $t \in \mathbb{R}_0^+$. Putting $J = [-M, M]$, we apply the singularity assumption (K.3) and we obtain $I_2 \to 0$ as $w \to +\infty$. The proof is now complete. ■

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