Pregroups: Models and Grammars

Wojciech Buszkowski
Faculty of Mathematics and Computer Science
Adam Mickiewicz University
Poznań Poland

Abstract

Pregroups, introduced in Lambek [12], are a generalization of partially ordered groups. In [5], we have proven several theorems on pregroups and grammars based on the calculus of free pregroups, in particular, the weak equivalence of these grammars and context-free grammars. In the present paper, we obtain further results of that kind. We consider left and right pregroups, study concrete left and right pregroups consisting of monotone functions on a poset and of monotone relations on a poset, and adjust the equivalence theorem to grammars based on left (right) pregroups.

1 Pregroups

A structure $(G, \leq, \cdot, 1)$ is called *a partially ordered monoid* (p.o. monoid), if $(G, \cdot, 1)$ is a monoid (i.e. a semigroup with unit 1), $(G, \leq)$ is a poset (i.e. a partially ordered set), and the following monotonicity condition holds:

$$\text{(MON) if } a \leq b \text{ then } ca \leq cb \text{ and } ac \leq bc,$$

for all $a, b, c \in G$.

A pregroup is defined as a structure $(G, \leq, \cdot, l, r, 1)$ such that $(G, \leq, \cdot, 1)$ is a p.o. monoid, and l, r are unary operations on G, satisfying the following inequalities:

$$a^l a \leq 1 \leq aa^l \text{ and } aa^r \leq 1 \leq a^r a, \tag{1}$$

for all $a \in G$. The element a^l (resp. a^r) is called *the left* (resp. *right*) *adjoint* of a.

Standard examples of pregroups are partially ordered groups (p.o. groups), i.e. structures $(G, \leq, \cdot, ^{\cup}, 1)$ such that $(G, \leq, \cdot, 1)$ is a p.o. monoid, and $^{\cup}$ is a unary operation on G, satisfying the equalities: $a^{\cup}a = 1 = aa^{\cup}$, for all $a \in G$. An extensive exposition of the theory of p.o. groups is given in Fuchs [8]. Clearly, in a p.o. group one can define $a^l = a^r = a^{\cup}$.

A pregroup G is said to be *commutative*, if $ab = ba$, for all $a, b \in G$. If G is commutative, then $a^l = a^r$, for all $a \in G$, and G is, actually, a p.o. group. Accordingly, the notion of a pregroup is a generalization of that of a p.o. group which can be innovatory for noncommutative structures only. A pregroup is said to be *proper*, if it is not a p.o. group (that means, $a^l \neq a^r$, for some element a).

Pregroups are related to residuated monoids which are abstract models for the Lambek calculus with empty antecedents [11, 4]. *A residuated monoid* is a structure $(G, \leq, \cdot, \backslash, /, 1)$ such that $(G, \leq, \cdot, 1)$ is a p.o. monoid, and $\backslash, /$ are binary operations on G, fulfilling the equivalences:

$$ab \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b, \tag{2}$$

for all $a, b, c \in G$. (Actually, (MON) follows from (2).) For a pregroup G, one defines $a\backslash b = a^r b$ and $a/b = ab^l$, which yields a residuated monoid. One can also define $a^l = 1/a$ and $a^r = a\backslash 1$. The latter definitions of adjoints, applied to an arbitrary residuated monoid, lead to inequalities $a^l a \leq 1$ and $aa^r \leq 1$, but the inequalities $1 \leq aa^l$ and $1 \leq a^r a$ need not hold.

The Lambek calculus with empty antecedents is a formal system whose theorems are the inequalities valid in residuated monoids. Consequently, all theorems of the Lambek calculus are also valid in pregroups under the above interpretation of residuals $a\backslash b$ and a/b. The converse is not true even for product-free types [5]. Grammars based on the Lambek calculus are a kind of categorial grammars, extensively studied nowadays as a framework for deductive parsing (see [14, 15, 3]). Lambek [12] proposes alternative parsing strategies, based on the calculus of free pregroups (also see [2, 7]). We discuss these matters in section 3.

Using (1), one easily proves the following equalities hold in every pregroup:

$$1^l = 1^r = 1,\ a^{lr} = a^{rl} = a,\ (ab)^l = b^l a^l,\ (ab)^r = b^r a^r. \tag{3}$$

By (1), we also derive:

$$aa^l a = a,\ aa^r a = a, \tag{4}$$

and (MON) with (1) yield:

$$\text{(MON')}\ a \leq b \text{ iff } b^l \leq a^l \text{ iff } b^r \leq a^r.$$

By $\mathbf{Z}$ we denote the set of integers. Following [12], for any pregroup G, $a \in G$, and $n \in \mathbf{Z}$, we define the element $a^{(n)} \in G$: $a^{(0)} = a$, $a^{(n+1)} = (a^{(n)})^r$, for $n \geq 0$, and $a^{(n-1)} = (a^{(n)})^l$, for $n \leq 0$. Using (3), one shows the latter equalities hold for all $n \in \mathbf{Z}$. Also:

$$a^{(n)} a^{(n+1)} \leq 1 \leq a^{(n+1)} a^{(n)}, \text{ for all } n \in \mathbf{Z}, \tag{5}$$

$$(ab)^{(n)} = a^{(n)} b^{(n)}, \text{ and if } a \leq b \text{ then } a^{(n)} \leq b^{(n)} \ (n \text{ is even}), \tag{6}$$

$$(ab)^{(n)} = b^{(n)} a^{(n)}, \text{ and if } a \leq b \text{ then } b^{(n)} \leq a^{(n)} \ (n \text{ is odd}). \tag{7}$$

Proposition 1 *Let G be a p.o. monoid. Then, for any $a \in G$, there exist at most one $b \in G$ such that $ba \leq 1 \leq ab$ and at most one $c \in G$ such that $ac \leq 1 \leq ca$.*

PROOF. Assume $ba \leq 1 \leq ab$ and $b'a \leq 1 \leq ab'$. Then $b \leq bab' \leq b'$ and $b' \leq b'ab \leq b$, hence $b = b'$. $\Box$

As a consequence, a^l is the only element b such that $ba \leq 1 \leq ab$, and a^r is the only element c such that $ac \leq 1 \leq ca$ (this yields an easy proof for (3)). Further, $()^l, ()^r, ()^{(n)}$ can be regarded as partial operations on arbitrary p.o. monoids, and they satisfy (1), (MON'), (3), (4), (5), (6), (7), if the relevant elements exist; for instance, if $a^{(n)}, b^{(n)}$ exist, then $(ab)^{(n)}$ exists and the equation from (6) holds. In a pregroup, right and left adjoints are uniquely determined by the underlying p.o. monoid. Accordingly, we can also define pregroups as p.o. monoids in which every element admits the two adjoints (then, the signature of pregroups equals that of p.o. monoids, and we may say, for example: a given pregroup is a substructure of a given p.o. monoid).

Proposition 2 *For every p.o. monoid G, there exists a (unique) largest pregroup which is a substructure of G.*

PROOF. Let H consist of all $a \in G$ such that $a^{(n)}$ exists, for all $n \in \mathbf{Z}$. By (6), (7), H is a submonoid of G. By the definition, H admits adjoints. Every pregroup H' which is a substructure of G must be contained in H. $\Box$

The pregroup H, defined above, will be denoted by $\mathrm{Pr}(G)$.

Let $(X, \leq)$ be a poset. A function $f : X \mapsto X$ is said to be *monotone*, if $x \leq y$ entails $f(x) \leq f(y)$. By $F(X, \leq)$ we denote the p.o. monoid of all monotone functions $f : X \mapsto X$ with the identity function I being the unit, the function composition being the operation $\cdot$, and the ordering being defined by: $f \leq g$ iff $f(x) \leq g(x)$, for all $x \in X$.

Proposition 3 *Every pregroup $(G, \leq, \cdot, l, r, 1)$ is embeddable into $Pr(F(G, \leq))$.*

PROOF. For $a \in G$, the function $f_a : G \mapsto G$ is defined as follows: $f_a(x) = ax$. By (MON), f_a is monotone. We have: $f_{ab} = f_a \circ f_b$, $f_1 = I$, and $a \leq b$ iff $f_a \leq f_b$. Consequently, the mapping $h(a) = f_a$ is a monomorphism of p.o. monoids. Since $f_{a^l} \circ f_a \leq I \leq f_a \circ f_{a^l}$ and $f_a \circ f_{a^r} \leq I \leq f_{a^r} \circ f_a$, then $f_{a^l} = f_a^l$ and $f_{a^r} = f_a^r$, hence h is a monomorphism of pregroups. $\Box$

Consequently, every pregroup is isomorphic to a pregroup of monotone functions on a poset.

If $(X, \leq)$ is a poset and $Y \subseteq X$, then $\max Y$ and $\min Y$ will denote the greatest and the least, respectively, element of Y (thus, $\max Y$ is the l.u.b. of Y belongimg to Y, and $\min Y$ is the g.l.b. of Y belonging to Y).

Proposition 4 *For every function $f \in Pr(F(X, \leq))$, there hold the equalities:*
$f^l(x) = min\{y \in X : x \leq f(y)\}$, for all $x \in X$,
$f^r(x) = max\{y \in X : f(y) \leq x\}$, for all $x \in X$.

Further, if M is a submonoid of $F(X, \leq)$ and, for every $f \in M$, the functions f^l, f^r defined by these equations exist and belong to M, then M is a pregroup.

PROOF. The first part is easy (see [5]). We prove the second part. Let M satisfy the condition. We prove (1). Since $f(x) \leq f(x)$, then $f^l(f(x)) \leq x$ (so, $f^l \circ f \leq I$). Also $x \leq f(f^l(x))$, since $f^l(x)$ is a y such that $x \leq f(y)$. □

These equalities can equivalently be expressed by the following Galois connections:

$$f^l(x) \leq y \text{ iff } x \leq f(y), \tag{8}$$

$$y \leq f^r(x) \text{ iff } f(y) \leq x, \tag{9}$$

for all $x, y \in X$.

As a consequence, all functions in $\Pr(F(X, \leq))$ must be upward and downward unbounded, that means:

$$\forall x \exists y (x \leq f(y)),\ \forall x \exists y (f(y) \leq x), \tag{10}$$

for all $x, y \in X$.

2 Left and right pregroups

A left (resp. *right*) *pregroup* is a p.o. monoid in which every element admits a left (resp. right) adjoint, that means, for every element a there exists an element a^l (resp. a^r) such that the left (resp. right) conjunct of (1) holds. Right pregroups are dual to left pregroups: $(G, \leq, \cdot, 1)$ is a left pregroup iff $(G, \geq, \cdot, 1)$ is a right pregroup. A pregroup is a p.o. monoid which is both a left pregroup and a right pregroup.

Everything which has been said above on pregroups can also be said on left (right) pregroups provided one confine herself to left (right) adjoints. In particular, every p.o. monoid G contains a largest left (resp. right) pregroup to be denoted by $\mathrm{LPr}(G)$ (resp. $\mathrm{RPr}(G)$); its elements are all $a \in G$ such that $a^{(n)}$ exists, for all $n \leq 0$ (resp. $n \geq 0$). Clearly, $\Pr(G)$ equals the meet of $\mathrm{LPr}(G)$ and $\mathrm{RPr}(G)$.

If we prove later on that no left (right) pregroup having a given property exists, then it follows that no pregroup having this property exists.

Of course, some arguments sound for pregroups need not be sound for left (right) pregroups. For example, to prove (MON') for pregroups one can prove: $a \leq b$ entails $b^l \leq a^l$ and $b^r \leq a^r$, and then, substitute b^l for a and a^l for b, and use the second equalities of (3) to obtain: $b^l \leq a^l$ entails $a \leq b$. For left pregoups, the latter entailment can be shown in another way. Assume $b^l \leq a^l$. Then:

$$a \leq bb^la \leq ba^la \leq b.$$

By an analogue of proposition 4, all functions in $\mathrm{LPr}(F(X,\leq))$ must be upward unbounded, and all functions in $\mathrm{RPr}(F(X,\leq))$ must be downward unbounded. These claims are dual to each other, since $\mathrm{LPr}(F(X,\leq))$ equals $\mathrm{RPr}(F(X,\geq))$.

Proposition 5 *Let a be the least (resp. greatest) element of poset $(X,\leq)$. Then, for every function $f \in RPr(F(X,\leq))$ (resp. $LPr(F(X,\leq))$) there holds: $f(x) = a$ iff $x = a$, for all $x \in X$.*

PROOF. Let $f \in \mathrm{RPr}(F(X,\leq))$. Then, $f(a) = a$, since f is downward unbounded. Assume $f(x) = a$. Then $a = f^r(a) \geq x$, by proposition 4, hence $x = a$. $\Box$

In [5], it has been shown that, for any totally ordered set $(X,\leq)$ containing more than two elements, $\mathrm{Pr}(F(X,\leq))$ consists of all both upward and downward unbounded, monotone functions $f : X \mapsto X$ if and only if $(X,\leq)$ is isomorphic to the set of integers with the natural ordering. We prove a left (right) analogue of this theorem.

Theorem 1 *Let $(X,\leq)$ be a totally ordered set containing at least two elements. Then, $LPr(F(X,\leq))$ consists of all upward unbounded, monotone functions $f : X \mapsto X$ if and only if $(X,\leq)$ is a well ordered set without the greatest element. Dually, $RPr(F(X,\leq))$ consists of all downward unbounded, monotone functions $f : X \mapsto X$ if and only if $(X,\geq)$ is a well ordered set without the greatest element.*

PROOF. Assume $(X,\leq)$ be a nonempty, well ordered set without the greatest element. Let M denote the set of all upward unbounded, monotone functions $f : X \mapsto X$. M is closed under function composition and $I \in M$, hence M is a submonoid of $F(X,\leq)$. Further, for any $f \in M$, the function f^l defined by the equation from proposition 4 exists and belongs to M (Caution: the nonexistence of the greatest element is needed to show that F^l is upward unbounded.) So, M is a left pregroup. But all functions in $\mathrm{LPr}(F(X,\leq))$ are upward unbounded, hence M equals the latter.

Now, let $(X,\leq)$ be a totally ordered set containing at least two elements. Assume $\mathrm{LPr}(F(X,\leq))$ consist of all upward unbounded, monotone functions $f : X \mapsto X$.

Suppose a to be the greatest element of X. Define $f(x) = a$, for all $x \in a$. Clearly, f is upward unbounded and monotone, hence $f \in \mathrm{LPr}(F(X,\leq))$, which contradicts proposition 5.

Suppose $(X,\leq)$ not to be well ordered. Then, there exists an infinite sequence $a_n \in X$, $n \geq 0$, such that $a_{n+1} < a_n$, for all n. Let Y denote the set of all $y \in X$ such that $a_{n+1} \leq y \leq a_n$, for some n. Define $f(x) = a_0$, for $x \in Y$, and $f(x) = x$, for $x \notin Y$. Clearly, f is upward unbounded and monotone. But $f^l(a_0) = \min(Y)$ does not exist. Contradiction. $\Box$

By ω we denote the set of nonnegative integers. It follows from theorem 1 that $\mathrm{LPr}(F(\omega,\leq))$, where $\leq$ is the natural ordering, contains all upward unbounded, monotone functions from ω into ω. It is easy to see that this left pregroup is proper (that means, it is not a p.o. group). Below we show how to do that.

An element a of a left (resp. right) pregroup is said to be *injective*, if $a^l a = 1$ (resp. $a^r a = 1$), and *surjective*, if $aa^l = 1$ (resp. $aa^r = 1$); is said to be *bijective*, if it is both injective and surjective. In a left pregroup of functions, a function f is injective (resp. surjective) iff it is an injective (resp. a surjective) mapping. We prove this fact. Assume f be injective. Then, $f^l \circ f = I$, so f is an injective mapping, since I is so. In a similar way, we show that, if f is surjective, then f is a surjective mapping. Assume f be an injective mapping. By (4), $f \circ f^l \circ f = f = f \circ I$, and consequently, $f^l \circ f = I$, so f is injective. In a similar way, we show that, if f is a surjective mapping, then f is surjective. The same holds true for right pregroups of functions.

Accordingly, in a left (right) pregroup of functions, a function f is bijective iff f is a bijective mapping. A left (right) pregroup is proper iff it contains at least one nonbijective element. Thus, $\mathrm{LPr}(F(\omega, \leq))$ is proper, since it contains nonbijective mappings, e.g. $f(x) = 2x$.

Proposition 6 *For any left pregroup G and all $a, b \in G$, the following conditions hold true:*

(i) a is injective (resp. surjective) iff a^l is surjective (resp. injective),

(ii) if ab is injective (resp. surjective), then b is injective (resp. a is surjective).

Proof. If $a^l a = 1$, then $a^l a^{ll} = 1$, and if $aa^l = 1$, then $a^{ll} a^l = 1$, by (3). The converse conditionals also hold, by (3) and (MON'). This proves (i). If $(ab)^l ab = 1$, then

$$1 = b^l a^l ab \leq b^l b \leq 1,$$

hence $b^l b = 1$, and if $ab(ab)^l = 1$, then

$$1 = abb^l a^l \geq aa^l \geq 1,$$

hence $aa^l = 1$, which proves (ii). □

An analogous fact is true for right pregroups.

Proposition 7 *If all elements of a left (right) pregroup are injective or surjective, then all elements are bijective.*

Proof. Assume that all elements of a left pregroup G be injective or surjective. Let $a \in G$. We prove that a is injective. Suppose not. Then, $a^l a$ is not injective, hence $a^l a$ is surjective, and consequently, a^l is surjective. So, a is injective. Contradiction. In a similar way, one proves that a is surjective. □

A left (right) pregroup is said to be *strictly proper*, if 1 is its only element which is injective or surjective. Every left (right) pregroup G contains a (unique) largest strictly proper left (right) pregroup G' which consists of all elements a of the former such that $a = 1$ or a is neither injective, nor surjective (by proposition

6, this set is closed under $\cdot$ and adjoints). If G is a pregroup, then G' is also a pregroup. By proposition 7, G is proper iff $G' \neq \{1\}$.

Theorem 2 *No totally ordered left (right) pregroup is proper.*

PROOF. We show: $a^l a = 1$ iff $a \geq a^{ll}$. First, $a^l a = 1$ iff $a^l a \geq 1$, by (1). If $a^l a \geq 1$, then $a^{ll} \leq a^{ll} a^l a \leq a$.If $a \geq a^{ll}$, then $a^l a \geq a^l a^{ll} \geq 1$. In a similar way, one shows: $aa^l = 1$ iff $a \leq a^{ll}$.

In a totally ordered left pregroup, $a \leq a^{ll}$ or $a^{ll} \leq a$, for every element a. Consequently, all elements are injective or surjective, and we use proposition 7. By duality, the same holds for right pregroups. □

It is known [8] that there are no finite p.o. groups in which $a < b$ holds, for some elements a, b. For $a < b$ entails $1 < ba^{\cup}$, and $1 < c$ entails $1 < c < cc < ccc < \ldots$. This argument cannot be applied for pregroups. We nonetheless prove:

Theorem 3 *If G is a finite left (right) pregroup, then the ordering in G is the identity relation (hence, G is a group).*

PROOF. Let $(G, \leq, \cdot, l, 1)$ be a finite left pregroup. The inequality $xa \leq b$ has a solution ($x = ba^l$), for all $a, b \in G$. For $a \in G$, define a mapping $f_a(x) = xa$. Let M denote the set of minimal elements of G. For any $b \in M$, there is $x \in G$ such that $f_a(x) = b$, hence there is $x \in M$ such that $f_a(x) = b$. Thus, f_a maps a subset of M onto M. Since M is finite, this subset equals M, and f_a restricted to M is a bijection of M onto itself. Consequently, $xa \in M$, for every $x \in M$.

The inequality $b \leq ax$ has a solution ($x = a^l b$), for all $a, b \in G$. For $a \in G$, define a mapping $g_a(x) = ax$. Let N denote the set of maximal elements of G. For any $b \in N$, there is $x \in G$ such that $g_a(x) = b$, hence there is $x \in N$ such that $g_a(x) = b$. As above, one shows that g_a restricted to N is a bijection of N onto itself. Consequently, $ax \in N$, for every $x \in N$.

Let $a \in N$. By the first paragraph, all minimal elements are of the form xa, for $x \in M$. By the second paragraph, these elements are maximal. We have shown that all minimal elements are maximal. Thus, the ordering is the identity relation. □

As a consequence, no finite left (right) pregroup can be proper. Below we obtain some results on the nonexistence of proper left or right pregroups of monotone functions on a poset.

Lemma 1 *Let $(X, \leq)$ be a poset, and $Y, Z \subseteq X$ be such that $X = Y \cup Z$, $Y \cap Z = \emptyset$. Let G be a left (right) pregroup contained in $F(X, \leq)$ such that, for all $f \in G$, $f[Y] \subseteq Y$ and $f[Z] \subseteq Z$. Let G_Y consist of all functions $f \in G$ restricted to Y, and G_Z consist of all $f \in G$ restricted to Z. Then, both G_Y and G_Z are left (right) pregroups contained in $F(Y, \leq)$ and $F(X, \leq)$, respectively (here, $\leq$ is the restricted ordering).*

PROOF. Clearly, both G_Y and G_Z are closed under composition. Since $I \in G$, then $I_Y \in G_Y$ and $I_Z \in G_Z$ (here, I_Y is the identity function restricted to Y, and similarly for I_Z). If $f \in G$, then $f^l \in G$, and consequently, f^l restricted to Y belongs to G_Y. The condition $f^l(f(x)) \leq x \leq f(f^l(x)$, for all $x \in X$, holds true in G, hence it also holds true in G_Y and G_Z for f and f^l restricted to Y and Z, respectively. So, G_Y and G_Z admit adjoints. □

The direct product $G \times H$ of left (right) pregroups G, H is defined in the usual way. If G, H are left (right) pregroups, then $G \times H$ is a left (right) pregroup. In lemma 1, G is embeddable into $G_Y \times G_Z$. G is isomorphic to $G_Y \times G_Z$, if either $y \leq z$, for all $y \in Y$, $z \in Z$, or neither $y \leq z$, nor $z \leq y$. for all $y \in Y$, $z \in Z$.

A poset $(X, \leq)$ is called *a tree*, if $\{y \in X : y \leq x\}$ is well ordered, for all $x \in X$. It is said to be *well founded*, if every nonempty subset of X has a minimal element. Every tree is well founded.

Theorem 4 *If $(X, \leq)$ is well founded, then no right pregroup contained in $F(X, \leq)$ is proper.*

PROOF. We fix a well founded poset $(X, \leq)$ and a right pregroup G contained in $F(X, \leq)$. It is sufficient to show that all functions in G are bijective.

We define a transfinite sequence of sets M_α. M_0 is the set of all minimal elements in $(X, \leq)$. For $\alpha > 0$, if $\bigcup_{\beta<\alpha} M_\beta$ is different from X, then M_α is the set of all minimal elements in $X - \bigcup_{\beta<\alpha} M_\beta$; otherwise our construction stops.

We show that, for all $f \in G$, f restricted to M_0 is a bijection of M_0 onto itself, and f restricted to $X - M_0$ maps $X - M_0$ into itself. First, we show $M_0 \subseteq f[M_0]$. Let $x \in M_0$. Since f is downward unbounded, then $f(y) = x$, for some $y \in X$. We take $z \in M_0$ such that $z \leq y$. Then, $f(z) \leq f(y)$, so $f(z) = x$. Second, we show $f[M_0] \subseteq M_0$. Let $x \in M_0$. Since f^r is downward unbounded, then we find $y \in M_0$ such that $f^r(y) = x$. Then, $f(x) = f(f^r(y)) \leq y$, so $f(x) \in M_0$. Third, we show $f[X - M_0] \subseteq X - M_0$. Assume $f(x) \in M_0$. Since $f^r[M_0] \subseteq M_0$ and $x \leq f^r(f(x))$, then $x \in M_0$. Finally, every $f \in G$ restricted to M_0 is one-to-one. Assume $f(x) = f(y)$, for $x, y \in M_0$. Denote $a = f^r(f(x))$. By the above, $a \in M_0$, and $x \leq a$, $y \leq a$, hence $x = y$.

Now, we can precisely formulate our inductive claim: for any α such that M_α is defined and for all $f \in G$, f restricted to M_α is a bijection of M_α onto itself, and f restricted to $X - \bigcup_{\beta\leq\alpha} M_\beta$ maps this set into itself. Assume this claim hold, for all $\gamma < \alpha$. Denote $Y = \bigcup_{\gamma<\alpha} M_\gamma$, $Z = X - Y$. Then, all functions $f \in G$ map bijectively Y onto Y and map Z into Z. By lemma 1, G_Z is a right pregroup contained in $F(Z, \leq)$. Since $(Z, \leq)$ is well founded, we can repeat the above arguments with G_Z in the place of G. Then, M_α takes the place of M_0. Consequently, all functions f in G_Z map bijectively M_α onto itself and restricted to $Z - M_\alpha$ map this set into itself. This proves our inductive claim.

Consequently, every $f \in G$ is a join of bijections on pairwise disjoint sets, hence it is a bijection. □

Actually, we can say more. Let f_α denote the restriction of f to M_α. For all $f, g \in G$, $f \leq g$ iff $f_\alpha \leq g_\alpha$, for all α, iff $f_\alpha = g_\alpha$, for all α, iff $f = g$. Consequently, every right pregroup of monotone functions on a well founded poset is a p.o. group whose ordering relation is the identity. Further, every well founded (right) pregroup is a p.o. group with the identity ordering (use theorem 4 and proposition 3), which yields a generalization of theorem 3.

$\mathrm{RPr}(F(\omega, \leq))$ contains the only element I, since I is the only monotone bijection on this tree; evidently, the largest pregroup contained in $F(\omega, \leq)$ is the same structure. Thus, the largest (left, right) pregroups contained in a p.o. monoid need not include all injective or surjective elements of the p.o. monoid. For the full binary tree $\{0,1\}^*$, the largest right pregroup of monotone functions equals the group of all monotone bijections; one of its elements is the function: $f(\epsilon) = \epsilon$, $f(0x) = 1x$, $f(1x) = 0x$, for $x \in \{0,1\}^*$.

Let $(X, \leq)$ be a totally ordered set. A function $f : X \mapsto X$ is said to be *downward continuous* (resp. *upward continuous*), if, for all $x \in X$, $f(x) =$l.u.b.$\{f(y) : y < x\}$ (resp. $f(x) =$g.l.b.$\{f(y) : x < y\}$). The set $(X, \leq)$ is said to be *dense*, if, for all $x, y \in X$ such that $x < y$ there exists $z \in X$ such that $x < z < y$.

Lemma 2 *If $(X, \leq)$ is a dense, totally ordered set, then $LPr(F(X, \leq))$ (resp. $RPr(F(X, \leq))$) contains upward (resp. downward) continuous functions only.*

PROOF. If $x < y$, then $f(x) \leq f(y)$, so $f(x)$ is a lower bound of $\{f(y) : x < y\}$. Assume $f(x)$ be not the g.l.b. of this set. Then, there exists z such that $f(x) < z$ and z is a lower bound of this set. The element $f^l(z)$ exists and equals $\min\{y : z \leq f(y)\}$. But $z \leq f(y)$ iff $x < y$, and the set $\{y : x < y\}$ has no least element. Contradiction. □

Theorem 5 *If $(X, \leq)$ is a dense, totally ordered set, then no left (right) pregroup contained in $F(X, \leq)$ is proper.*

PROOF. Let G be a left pregroup contained in $F(X, \leq)$, and let $f \in G$. We show that f is injective. Assume $x < y$. Suppose $f(x) = f(y)$. Denote $a = f(y)$. By proposition 5, a is not the greatest element of X. We have $f^l(a) = f^l(f(x)) \leq x$, but, for all $z > a$, $f^l(z) =\min\{u : z \leq f(u)\} > y$. Thus, f^l is not upward continuous, which contradicts lemma 2. Accordingly, all functions in G are injective, and we use proposition 7. □

A poset $(X, \leq)$ is called *a lattice*, if, for all $a, b \in X$, there exist the g.l.b. and the l.u.b. of $\{a, b\}$; these elements will be denoted by $a \wedge b$ and $a \vee b$, respectively. A (left, right) pregroup is said to be *lattice ordered* (l.o.), if its underlying poset is a lattice. It is easy to show that in every l.o. pregroup the following equalities are valid:

$$(a \vee b)^l = a^l \wedge b^l,\ (a \wedge b)^l = a^l \vee b^l, \tag{11}$$

$$(a \vee b)^r = a^r \wedge b^r,\ (a \wedge b)^r = a^r \vee b^r, \tag{12}$$

$$(a \vee b)c = ac \vee bc,\ a(b \vee c) = ab \vee ac,\ (a \wedge b)c = ac \wedge bc,\ a(b \wedge c) = ab \wedge ac. \tag{13}$$

This is not the case for l.o. left (right) pregroups. On the other hand, if G is a p.o. monoid whose underlying poset is a lattice such that (13) are valid in G, then $\mathrm{LPr}(G)$ (resp. $\mathrm{RPr}(G)$) is a l.o. left (resp. right) pregroup satisfying (11) (resp. (12)). We omit the proof. Examples can be found as follows. Take a totally ordered set $(X, \leq)$. Then, $F(X, \leq)$ is a lattice with $(f \vee g)(x) = f(x) \vee g(x)$, $(f \wedge g)(x) = f(x) \wedge g(x)$, and it satisfies (13). So, the largest left (resp. right) pregroup contained in $F(X, \leq)$ is a l.o. left (resp. right) pregroup satisfying (11) (resp. (12)). Accordingly, there exist proper l.o. (left, right) pregroups, e.g. the largest pregroup contained in $F(\mathbf{Z}, \leq)$ and the largest left pregroup contained in $F(\omega, \leq)$. We show that pregroups cannot be bounded.

Proposition 8 *If a left (right) pregroup contains $\perp$ or $\top$, then it is trivial.*

PROOF. For pregroups the proof is a bit simpler than for left pregroups. Let G be a pregroup containing the least element $\perp$. For all $a \in G$, we have $\perp \leq a$, so $a^l \leq \perp^l$, and consequently, $a = (a^r)^l \leq \perp^l$. So, $\perp^l = \top$ is the greatest element of G. We have $\perp\perp \leq \perp$ and $\top \leq \top\top$, hence $\perp\perp = \perp$ and $\top\top = \top$. Also $\top^l = \perp$. Then, $\perp\top = \perp\perp\top = \perp\top^l\top \leq \perp$, so $\perp\top = \perp$, and $\perp\top = \perp\top\top = \perp\perp^l\top \geq \top$, so $\perp\top = \top$. Consequently, $\perp = \top$, hence G is trivial.

Now, let G be a left pregroup containing $\perp$. Again $a^l \leq \perp^l$, for all $a \in G$, but now we cannot infer $a \leq \perp^l$, for all $a \in G$. As above, $\perp\perp = \perp$, hence, by (3), $\perp^l\perp^l = \perp^l$. We have $\perp \leq \perp^{ll}$, so $\perp$ is surjective (see the proof of theorem 2) and $\perp\perp^l = 1$. Then, $\perp^l = \perp\perp^l\perp^l = \perp\perp^l = 1$, hence $\perp = \perp\perp^l = 1$. Since $a^l \leq \perp^l$, for all $a \in G$, then $a^l \leq \perp$, so $a^l = \perp$, for all $a \in G$. Consequently, $a^l = b^l$, which yields $a = b$, for all $a, b \in G$ (use (MON')).

In both cases, if G contains the greatest element $\top$, then the reasoning is dual. $\square$

3 Pregroups of relations

It has been proven in [1, 6] that every residuated monoid is embeddable into the residuated monoid of binary relations on some set X. The latter is defined as follows. For $R, S \subseteq X^2$, one defines:

$$R \circ S = \{(x, y) : \exists z((x, z) \in R \text{ and } (z, y) \in S)\},$$

$$R \backslash S = \{(x, y) : \forall z((z, x) \in R \Rightarrow (z, y) \in S)\},$$

$$S / R = \{(x, y) : \forall z((y, z) \in R \Rightarrow (x, z) \in S)\}.$$

$P(W)$ denotes the powerset of set W. $(P(X^2), \subseteq, \circ, \backslash, /, I)$ is a residuated monoid: the residuated monoid of binary relations on X.

Algebras of relations are important for many areas of logic in computer science (logics and semantics of programs, knowledge representation, logics in AI) [16]. In [10], residuated monoids of binary relations are used to model the notions of a weak prespecification and postspecification of programs (the underlying logic is the Lambek calculus with empty antecedents, not explicitly quoted in [10]).

In this section we examine algebras of binary relations as models of pregroups. First, we observe that in order to model proper pregroups we must modify the basic monoid of binary relations. The p.o. monoid $(P(X^2), \subseteq, \circ, I)$ contains no proper left (right) pregroup. Consider the condition (1):

$$R^l \circ R \subseteq I \subseteq R \circ R^l.$$

By $R^{\cup}$ we denote the converse of R. By the second inclusion of (1), X equals both the domain of R and the codomain of R^l. We prove $R^{\cup} \subseteq R^l$. Assume $(y, x) \in R^{\cup}$. Then $(x, y) \in R$. So, there is z such that $(z, x) \in R^l$. By the first inclusion of (1), $z = y$, hence $(y, x) \in R^l$. We prove $R^l \subseteq R^{\cup}$. Assume $(x, y) \in R^l$. Then, there is z such that $(y, z) \in R$. By the first inclusion of (1), $x = z$, hence $(y, x) \in R$ and $(x, y) \in R^{\cup}$. We have shown $R^l = R^{\cup}$. Thus, in every left pregroup contained in the standard p.o. monoid the left adjoint of R equals the converse of R. Then, from $(x, y) \in R$ and $(x, y') \in R$ it follows that $y = y'$, again by the first inclusion of (1), so each relation R in this left pregroup is a function. But $R^{\cup}$ must also be a function, so R is a bijection from its domain (i.e. X) onto its codomain (again X, since X is the domain of $R^{\cup}$).

Let $(X, \leq)$ be a poset. A relation $R \subseteq X^2$ is said to be *monotone*, if it satisfies the following conditions: (i) xRy and $x' \leq x$ entail $x'Ry$, (ii) xRy and $y \leq y'$ entail xRy'. (We write xRy for $(x, y) \in R$.) By $R(X, \leq)$ we denote the set of all monotone relations $R \subseteq X^2$. This set is a p.o. monoid with the ordering $\leq$, the operation $\circ$ and the unit element $\leq$.

Proposition 9 *Every (left, right) pregroup $(G, \leq, \circ, 1)$ is embeddable into the largest (left, right) pregroup contained in $R(G, \leq)$.*

PROOF. Fix a (left, right) pregroup $(G, \leq, \circ, 1)$. For $a \in G$, define $R_a \subseteq G^2$ by: xR_ay iff $x \leq ay$. Clearly R_a is monotone. If $xR_{ab}y$, then, for $z = by$, we have xR_az and zR_by, so $R_{ab} \subseteq R_a \circ R_b$. Assume $xR_a \circ R_by$. Then, there exists z such that $x \leq az$ and $z \leq by$, hence $x \leq aby$. We have shown $R_{ab} = R_a \circ R_b$. Clearly $R_1 = \leq$. Also $a \leq b$ iff $R_a \subseteq R_b$ (from the right to the left: if $R_a \subseteq R_b$, then aR_a1 implies aR_b1 which means $a \leq b$). Now, (1) hold in G, which yields:

$$R_{a^l} \circ R_a \subseteq \leq \subseteq R_a \circ R_{a^l} \text{ and } R_a \circ R_{a^r} \subseteq \leq \subseteq R_{a^r} \circ R_a,$$

for all $a \in G$. Consequently, R_{a^l} is the left adjoint of R_a, and R_{a^r} is the right adjoint of R_a. Then, the mapping $F(a) = R_a$ is the required embedding. □

The same construction can be performed for residuated monoids. Yet, in general, $R_{a \backslash b} \subseteq R_a \backslash R_b$ but the equality does not hold, and similarly for $/$.

Proposition 10 *Let G be a pregroup contained in $R(X, \leq)$. Then, for all $R \in G$ and all $x, y \in X$, the following equivalences hold true:*

$$xR^l y \Leftrightarrow \forall z(yRz \Rightarrow x \leq z),\ xR^r y \Leftrightarrow \forall z(zRx \Rightarrow z \leq y).$$

Proof. We prove the first equivalence. Since $R^l \circ R \subseteq \leq$, then:

$$xR^l y \Rightarrow \forall z(yRz \Rightarrow x \leq z).$$

Since $\leq \subseteq R \circ R^l$, then:

$$x \leq y \Rightarrow \exists z(xRz \text{ and } zR^l y.$$

Since $x \leq x$, then there exists z such that xRz and $zR^l x$, hence, for all u, if xRu then $z \leq u$. So, $z = \min\{u : xRu\}$. We have shown that, for all $x \in X$ there exists the least u such that xRu; this element u will be denoted by x_R. We have xRx_R and $x_R R^l x$. Assume $\forall z(yRz \Rightarrow x \leq z)$. Since yRy_R, then $x \leq y_R$, and since $y_R R^l y$, then $xR^l y$ (R^l is monotone). We have proven the first equivalence.

The proof of the second equivalence is dual. Now, one shows that, for all $x \in X$, there exists the greatest element u such that uRx; this element u will be denoted by x^R. We have $x^R Rx$ and $xR^r x^R$. □

Since the left and the right part of this proof are independent of each other, then an analogous result holds for left (right) pregroups. The elements x_R and x^R defined in the proof satisfy the following equivalences:

$$xRy \text{ iff } x_R \leq y \text{ iff } x \leq y^R, \tag{14}$$

for all $x, y \in X$.

We have: $x \leq y$ entails $x_R \leq y_R$ and $x^R \leq y^R$. Also:

$$R \subseteq S \text{ iff } \forall x(x_S \leq x_R) \text{ iff } \forall x(x^R \leq x^S), \tag{15}$$

$$(x_R)_S = x_{R \circ S},\ (x^R)^S = x^{S \circ R}. \tag{16}$$

We prove the second equality of (16). Assume $z \leq (x^R)^S$. Then, zSx^R, by (14), and $x^R Rx$, so $zS \circ Rx$. which yields $z \leq x^{S \circ R}$, by (14). Assume $z \leq x^{S \circ R}$. By (14), $zS \circ Rx$, hence there is u such that zSu and uRx. By (14), $u \leq x^R$, so zSx^R, which yields $z \leq (x^R)^S$, by (14).

Denote $G = \Pr(R(X, \leq))$. To every relation $R \in G$ we assign a function $f^R : X \mapsto X$ by setting: $f^R(x) = x^R$. Clearly, f^R is monotone. By (14), $f^{\leq} = I$. By (16), $f^{S \circ R} = f^S \circ f^R$. By (15), $R \subseteq S$ iff $f^R \leq f^S$. Since (1) hold in G, then:

$$f^{R^l} \circ f^R \leq I \leq f^R \circ f^{R^l},\ f^R \circ f^{R^r} \leq I \leq f^{R^r} \circ f^R,$$

hence $f^{R^l} = (f^R)^l$ and $f^{R^r} = (f^R)^r$. Consequently, the mapping $F(R) = f^R$ is a monomorphism of G into $\Pr(F(X, \leq))$. Actually, F is an isomorphism. For the mapping $F'(f) = R_f$, where the relation $R_f \subseteq X^2$ is defined by: xR_fy iff $x \leq f(y)$, is a monomorphism of $\mathrm{Pr}(F(X, \leq))$ into G converse to F. We omit an easy proof of this fact. Analogous arguments work for right pregroups. Accordingly, the following theorem is true.

Theorem 6 *For every poset $(X, \leq)$, the largest (resp. right) pregroup contained in $R(X, \leq)$ is isomorphic to the largest (resp. right) pregroup contained in $F(X, \leq)$.*

Interestingly, there is no full symmetry between the left and the right case. Denote $G = \mathrm{LPr}(R(X, \leq))$ and $H = \mathrm{LPr}(F(X, \leq))$. For $R \in G$, define: $f_R(x) = x_R$, for $x \in X$. Again, f_R is monotone. However, $R \subseteq S$ iff $f_S \leq f_R$, and $f_{S \circ R} = f_R \circ f_S$, by (15) and (16). One can show that the mapping $F(R) = f_R$ is a $\circ$ and $\leq$ inverting isomorphism of G onto H (it preserves left adjoints).

We show that, if $(X, \leq)$ is the poset $(\omega, \leq)$, then no isomorphism (i.e. $\leq$ and $\circ$ and l preserving bijection) from G onto H exists. For assume it exist. Then, by the composition of this morphism with the converse of F from the above paragraph, we obtain a $\circ$ and $\leq$ inverting and l preserving isomorphism h of H onto itself. But $g(f) = f^l$ defines a $\circ$ and $\leq$ inverting and l preserving monomorphism of H into itself (use (3) and (MON')). Denote $H' = g(H)$. Clearly, $g \circ h$ is an isomorphism of H onto H'. H' consists of all f^l, for $f \in H$. For every p.o. monoid, elements a^l are precisely those elements which admit right adjoints in this p.o. monoid. A function $f \in H$ admits a right adjoint iff f is downward unbounded, which is equivalent to the condition $f(0) = 0$. Now, all surjective elements of H satisfy this condition, hence they admit right adjoints in H. On the other hand, there are surjective elements in H' which do not admit right adjoints in H', e.g. $f(0) = 0$, $f(x) = x - 1$, for $x \neq 0$ (then, $f^r(0) = 1$, so $f^r \notin H'$). Consequently, H is not isomorphic to H', which falsifies our assumption. Observe that G is isomorphic to H'.

4 Grammars

In [5], it has been proven that grammars based on free pregroups are weakly equivalent to context-free grammars. In this section, we outline a proof of an analogous theorem for grammars based on free left (right) pregroups.

Let $(X, \leq)$ be a poset. *Terms* are expressions $a^{(n)}$, for $a \in X$, $n \in \mathbf{Z}$. *Types* are finite strings of terms; ϵ denotes the empty type.For types x, y, the concatenation of x and y is denoted by xy. The relation $\leq$ on the set of types is defined as the smallest reflexive and transitive relation, satisfying the following conditions: (CON) $xa^{(n)}a^{(n+1)}y \leq xy$, (EXP) $xy \leq xa^{(n+1)}a^{(n)}y$, (IND) $xa^{(n)}y \leq xb^{(n)}y$, if either $a \leq b$ and n is even, or $b \leq a$ and n is odd, for all $a, b \in X$, types x, y and $n \in \mathbf{Z}$. We write $x \sim y$ if $x \leq y$ and $y \leq x$. Equivalence classes of $\sim$ are the members of *the free pregroup* generated by $(X, \leq)$. In this free pregroup, $1 = [\epsilon]$, $[x] \cdot [y] = [xy]$, $[x] \leq [y]$ iff $x \leq y$, and the adjoints of $[x] \neq 1$ are defined as follows:

$$[a_1^{(n_1)} \dots a_k^{(n_k)}]^l = [a_k^{(n_k-1)} \dots a_1^{(n_1-1)}],$$

$$[a_1^{(n_1)} \dots a_k^{(n_k)}]^r = [a_k^{(n_k+1)} \dots a_1^{(n_1+1)}].$$

This construction is due to Lambek [12]. (CON), (EXP) and (IND) can be treated as rewriting rules of a formal calculus: $x \leq y$ holds iff one can rewrite x into y by a finite number of applications of these rules. Lambek introduces more general rules: (GCON) $xa^{(n)}b^{(n+1)}y \leq xy$, (GEXP) $xy \leq xa^{(n+1)}b^{(n)}y$, both requiring that either n is even and $a \leq b$ in X, or n is odd and $b \leq a$ in X. Clearly, (GCON) is (IND) followed by (CON), and (GEXP) is (EXP) followed by (IND). A key result in [12] is the following Switching Lemma: if $x \leq y$, then there exist types u, v such that $x \leq u$ by (GCON) only, $u \leq v$ by (IND) only, and $v \leq y$ by (GEXP) only. Consequently, if $x \leq y$ and y is a term or $y = \epsilon$, then $x \leq y$ by (GCON) and (IND) only, that means, by (CON) and (IND) only.

As a consequence of the Switching Lemma, we show that free pregroups generated by posets are strictly proper. If $[x]$ is injective (resp. surjective), where $x = x_1 \dots x_k$, $x_1, \dots, x_k$ are terms, then $[x_k]$ is injective (resp. $[x_1]$ is surjective). Thus, it is sufficient to show that there exists no term x such that $[x]$ is injective or surjective. Take a term $a^{(n)}$. Assume $[a^{(n)}]$ be injective. Then $[a^{(n-1)}a^{(n)}] = 1$, so $\epsilon \leq a^{(n-1)}a^{(n)}$. By the Switching Lemma, the latter can be derived by (IND) and (GEXP) only, which is impossible. In a similar way, one shows that $[a^{(n)}]$ cannot be surjective.

By *a pregroup grammar* we mean a quadruple $(V, X, \leq, R)$ such that V is a nonempty, finite alphabet, $(X, \leq)$ is a finite poset, and R is a finite relation betwen elements of V and types on $(X, \leq)$. One says that the grammar assigns type x to string $v_1 \dots v_n$, $v_i \in V$, if there exists types x_i such that $v_i R x_i$, for $i = 1, \dots, n$, satisfying: $x_1 \dots x_n \leq x$ in the above sense. If G is a pregroup grammar and x is a type, then $L(G, x)$ denotes the set of all strings on V which are assigned type x by G. Using Switching Lemma, one proves that, for every pregroup grammar G and every term x, $L(G, x)$ is a context-free language (not containing ϵ). Conversely, for every context-free language L such that $\epsilon \notin L$, there exist a pregroup grammar G (whose poset $(X, \leq)$ is $(X, =)$) and a term $a \in X$ such that $L(G, a) = L$. Consequently, the languages generated by pregroup grammars are precisely the context-free languages not containing ϵ. Proofs of these results are given in [5].

Free left (resp. right) pregroups are defined as free pregroups except that one only admits terms $a^{(n)}$, for $n \leq 0$ (resp. $n \geq 0$) and drops right (resp. left) adjoints. It can easily be shown that free left (resp. right) pregroups are left (resp. right) pregroups. Further, if G is the free left (resp. right) pregroup generated by $(X, \leq)$ and H is an arbitrary left (resp. right) pregroup, then every $\leq$ preserving mapping h of X into H can uniquely be extended to a homomorphism of G into H, Consequently, $x \leq y$ (equivalently: $[x] \leq [y]$) holds G iff $x \leq y$ is true in every left (resp. right) pregroup H under every $\leq$ preserving assignment of members of H to elements of X.

If x is a type and y is a term, both in the sense of free left (resp. right) pregroups, then the following equivalence holds true: $x \leq y$ in the sense of free pregroups iff $x \leq y$ in the sense of free left (resp. right) pregroups. The conditional ($\Leftarrow$) is obvious, since the calculus of free pregroups is stronger than that of free left (resp. right) pregroups. To prove ($\Rightarrow$) assume $x \leq y$ in the sense of free pregroups. By Switching Lemma, $x \leq y$ by (CON) and (IND) only, and these steps can be performed in the calculus of free left (resp. right) pregroups, since they do not introduce any bad terms.

Grammars based on left (right) pregroups are defined as pregroup grammars except that one only admits types and rules in the sense of free left (right) pregroups. By the preceding paragraph and the results from [5], mentioned above, if G is a grammar based on left (right) pregroups and x is a term, then $L(G,x)$ is a context-free language. Conversely, if L is a context-free language such that $\epsilon \notin L$, then the pregroup grammar for L constructed in [5] involves left adjoints only (corresponding to types $p, p/q, (p/q)/r$), hence, by the preceding paragraph, it generates L as a grammar based on left pregroups. Also, one could use a grammar involving right adjoints only (corresponding to types $p, q\backslash p, r\backslash(q\backslash p)$) and construct a grammar based on right pregroups for L. Accordingly, grammars based on left (right) pregroups generate precisely the context-free languages not containing ϵ.

The same results can also be proven in a different, though essentially similar way. First, Switching Lemma can be proven for free left (right) pregroups by the argument from [12] (also see [5]). Further, the construction of a pregroup grammar for a given context-free language L, provided in [5], actually yields a grammar based on left (right) pregroups. Thus, one directly proves the weak equivalence of context-free grammars and grammars based on left (right) pregroups. Now, we have to prove the weak equivalence of grammars based on right pregroups and pregroup grammars. Switching Lemma for free pregroups implies that, for clauses $x \leq y$ such that x is a string of right terms and y is a right term, the calculus of free pregroups is conservative over the calculus of free right pregroups. Thus, every grammar based on right pregroups can be treated as a pregroup grammar. Conversely, given a pregroup grammar G, we define m to be the least $n \in \mathbf{Z}$ such that a term of the form $a^{(n)}$ appears in types involved in the relation R of G. If $m \geq 0$, then G can be treated as a grammar based on right pregroups. Consider the case $m < 0$. Define a mapping $f(a) = [a^{(-2m)}]$ from X into the free pregroup underlying G. By (MON'), f preserves $\leq$, so it can uniquely be extended to a homomorphism of this pregroup into itself. One easily proves that $f([x]) = [x]^{(-2m)}$, for all types x, hence f is an automorphism of this pregroup. Consequently, $[x] \leq [y]$ iff $f([x]) \leq f([y])$, for all types x, y. A grammar G' is constructed from G by replacing in R each $a^{(n)}$ by $a^{(n-2m)}$. By the above, for every term $a^{(n)}$, $L(G, a^{(n)}) = L(G', a^{(n-2m)})$. Yet, G' does not involve left terms, hence it can be treated as a grammar based on right pregroups. Therefore, grammars based on free right pregroups generate the same languages as pregroup grammars.

References

[1] H. Andréka and S. Mikulaś, Lambek calculus and its relational semantics: completeness and incompleteness, *Journal of Logic, Language and Information* 3 (1994), 1-37.

[2] D. Bargelli and J. Lambek, An algebraic approach to French sentence structure, in [9], 62-78.

[3] W. Buszkowski, Mathematical Linguistics and Proof Theory, in [17], 683-736.

[4] W. Buszkowski, Algebraic structures in categorial grammar, *Theoretical Computer Science* 199 (1998), 5-24.

[5] W. Buszkowski, Lambek grammars based on pregroups, in [9], 95-109.

[6] W. Buszkowski and M. Kołowska-Gawiejnowicz, Representation of residuated semigroups in some algebras of relations. (The method of canonical models.), *Fundamenta Informaticae* 31 (1997), 1-12.

[7] C. Casadio and J. Lambek, An algebraic analysis of clitic pronouns in Italian, in [9], 110-124.

[8] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, Oxford, 1963.

[9] P. de Groote, G. Morrill and C. Retoré (eds.), *Logical Aspects of Computational Linguistics*, LNAI 2099, Springer, Berlin, 2001.

[10] C. Hoare and H. Jifeng, The weakest prespecification, *Fundamenta Informaticae* 9 (1986), 51-84, 217-252.

[11] J. Lambek, The mathematics of sentence structure, *American Mathematical Monthly* 65 (1958), 154-170.

[12] J. Lambek, Type grammars revisited, in [13], 1-27.

[13] A. Lecomte, F. Lamarche and G. Perrier (eds.), *Logical Aspects of Computational Linguistics*, LNAI 1582, Springer, Berlin, 1999.

[14] M. Moortgat, *Categorial Investigations. Logical and Linguistic Aspects of the Lambek Calculus*, Foris, Dordrecht, 1988.

[15] M. Moortgat, Categorial Type Logics, in [17], 93-177.

[16] E. Orłowska and A. Szałas (eds.), *Relational Methods for Computer Science Applications*, Physica Verlag, Heidelberg, 2001.

[17] J. van Benthem and A. ter Meulen (eds.), *Handbook of Logic and Language*, Elsevier, Amsterdam, The MIT Press, Cambridge Mass., 1997.