Relational Models of Lambek Logics

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Abstract

Lambek logics are substructural logics related to the Syntactic Calculus of Lambek [17]. In this paper we prove several representation theorems for algebras of Lambek logics (residuated semigroups, residuated monoids and others) with respect to certain algebras of binary relations. First results of that kind have been obtained in [1], by a method of labelled graphs. By a method of labelled formulas, other results have been proven in [9, 28]. In the present paper, we prove these and other results by a construction of chains of partial representations; its idea has earlier been announced in abstract [5]. We also mention a simpler construction.

1 Lambek logics and their models

Substructural logics are logics whose Gentzen-style axiomatization abandons some (all) structural rules: Contraction, Weakening, Exchange ,etc.; a good introduction into this area is Restall [27]. Historically, one of the earliest substructural logics is the Syntactic Calculus of Lambek [17]: a type processing system for categorial grammars. Other well known representatives are systems of relevant logic. Models and proof techniques for substructural logics have been studied by numerous authors; see e.g. [11, 13, 10, 22, 24]. Models of Lambek logics have been studied in e.g. [3, 2, 12, 25, 1, 29, 16, 9, 6]; see also survey papers [4, 21].

A standard application of relational methods in modelling substructural logics are Kripke frames: an $n-$ary logical connective is interpreted by means of an $n+1-$ary relation on the universe. This kind of models is very popular and will not be considered here. We shall be interested in models whose elements are binary relations on a fixed universe, and logical connectives are interpreted by means of basic operations on binary relations. Models of that kind are significant for at least two reasons. First, they show a link between substructural logics and dynamic algebras. An early application of Lambek logics (essentially, but not explicitly) to modelling program specification is [15]; another example is Pratt [26], where residuation operations, characteristic to substructural logics, affixed to regular operations result in an elegant equational axiomatization of action logic. Second, they allow to interpret substructural logics within Tarski's relational calculus, which makes it possible to employ the latter as a proof system for the former; this line has been developed by Orłowska [23, 24, 20].

We describe the Syntactic Calculus, further called *Associative Lambek Calculus* (AL). Its formulas (sometimes called: types) are formed out of denumerably many variables (atoms) $p, q, r, \ldots$ and binary logical connectives $\otimes$ (product), $\rightarrow$ (right arrow) and $\leftarrow$ (left arrow). A, B, C denote formulas, and Γ, Δ denote finite strings of formulas. Sequents are expressions $\Gamma \vdash A$. Axioms are the sequents:

$$\text{(Id) } A \vdash A, \ \text{(AS) } (A \otimes B) \otimes C \vdash A \otimes (B \otimes C), \ A \otimes (B \otimes C) \vdash (A \otimes B) \otimes C,$$

and the inference rules are:

$$(\otimes\mathrm{L})\ \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, A \otimes B, \Delta \vdash C},\ (\otimes\mathrm{R})\ \frac{\Gamma \vdash A;\ \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B},$$

$$(\leftarrow\mathrm{L})\ \frac{\Gamma, B, \Gamma' \vdash C;\ \Delta \vdash A}{\Gamma, B \leftarrow A, \Delta, \Gamma' \vdash C},\ (\leftarrow\mathrm{R})\ \frac{\Gamma, A \vdash B}{\Gamma \vdash B \leftarrow A},$$

$$(\rightarrow\mathrm{L})\ \frac{\Gamma, B, \Gamma' \vdash C;\ \Delta \vdash A}{\Gamma, \Delta, A \rightarrow B, \Gamma' \vdash C},\ (\rightarrow\mathrm{R})\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B},$$

$$(\mathrm{CUT})\ \frac{\Gamma, A, \Gamma' \vdash B;\ \Delta \vdash A}{\Gamma, \Delta, \Gamma' \vdash B}.$$

In rules ($\leftarrow$R) and ($\rightarrow$R), one assumes $\Gamma \neq \epsilon$ (by ϵ we denote the empty string). Lambek [17] proves the cut elimination theorem for AL: every provable sequent has a proof without (CUT). Notice that $\Gamma \neq \epsilon$, for every provable sequent $\Gamma \vdash A$. We write $\Gamma \vdash_{AL} A$ if $\Gamma \vdash A$ is provable in AL, and similarly for other systems.

We also consider *Nonassociative Lambek Calculus* (NL), introduced in [18]. The formulas are the same as for AL. Formula structures (structures, for short) are defined by recursion: (i) all formulas are (atomic) structures, (ii) if X, Y are structures, then $(X \circ Y)$ is a structure. $X[Y]$ denotes a structure X with a designated occurrence of a substructure Y, and in this context $X[Z]$ denotes the result of replacing Y with Z in X. X, Y, Z range over structures. Sequents are expressions $X \vdash A$. The axioms are (Id), and the inference rules are:

$$(\otimes\mathrm{L})\ \frac{X[A \circ B] \vdash C}{X[A \otimes B] \vdash C},\ (\otimes\mathrm{R})\ \frac{X \vdash A;\ Y \vdash B}{X \circ Y \vdash A \otimes B},$$

$$(\leftarrow\mathrm{L})\ \frac{X[B] \vdash C;\ Y \vdash A}{X[(B \leftarrow A) \circ Y] \vdash C},\ (\leftarrow\mathrm{R})\ \frac{X \circ A \vdash B}{X \vdash B \leftarrow A},$$

$$(\rightarrow\mathrm{L})\ \frac{X[B] \vdash C;\ Y \vdash A}{X[Y \circ (A \rightarrow B)] \vdash C},\ (\rightarrow\mathrm{R})\ \frac{A \circ X \vdash B}{X \vdash A \rightarrow B},$$

$$(\mathrm{CUT})\ \frac{X[A] \vdash B;\ Y \vdash A}{X[Y] \vdash B}.$$

Again, the cut elimination theorem holds for NL [18]. NL admits no structural rules; it is regarded as the basic substructural logic in [21]. AL is equivalent to NL enriched with the structural rule of associativity: from $X[(Y_1 \circ Y_2) \circ Y_3] \vdash A$ infer $X[Y_1 \circ (Y_2 \circ Y_3)] \vdash A$, and conversely.

A natural extension of AL is AL0 which admits $\Gamma = \epsilon$ in rules ($\leftarrow$R) and ($\rightarrow$R). We have $\vdash_{AL0} p \rightarrow p$, by (Id) and ($\rightarrow$R), hence $(p \rightarrow p) \rightarrow p \vdash_{AL0} p$, by (Id) and ($\rightarrow$L), but the latter sequent is not provable in AL. So, AL0 is a nonconservative extension of AL. NL0 admits the empty structure 0 which satisfies $0 \circ X = X \circ 0 = X$, for every X. By the same example, NL0 is a nonconservative extension of NL. Strictly related systems are AL1 and NL1 which admit a logical constant 1. AL1 is AL0 with 1, the new axiom $\vdash 1$, and the new inference rule:

$$\text{(1L)}\ \frac{\Gamma, \Delta \vdash A}{\Gamma, 1, \Delta \vdash A}.$$

NL1 is NL0 with 1, the new axiom $0 \vdash 1$, and the new inference rule:

$$\text{(1L)}\ \frac{X[0] \vdash A}{X[1] \vdash A}.$$

The cut elimination theorem holds for each of these systems, and consequently, AL1 and NL1 are conservative extensions of AL0 and NL0, respectively.

The structural rule of exchange for associative systems can be written:

$$\text{(EXC)}\ \frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C},$$

and for nonassociative systems:

$$\text{(NEXC)}\ \frac{X[Y \circ Z] \vdash A}{X[Z \circ Y] \vdash A}.$$

AL with (EXC) is denoted by ALE, and AL0E, AL1E are used in a similar sense. NLE is NL with (NEXC), and similarly for NL0E, NL1E. We say that formulas A, B are *deductively equivalent* in system S, if both $A \vdash_S B$ and $B \vdash_S A$. One easily shows that $A \rightarrow B$ and $B \leftarrow A$ are deductively equivalent in all systems with exchange, mentioned above, hence $\leftarrow$ or $\rightarrow$ can be removed from the language. The product-free fragment of AL0E is equivalent to the logic BCI whose Hilbert-style axiomatization consists of the axioms:

(B) $(B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$,

(C) $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$,

(I) $A \rightarrow A$,

and the rule Modus Ponens: from $A \rightarrow B$ and A infer B. AL0E is a conservative fragment of Intuitionistic Linear Logic [14]. As shown in [6], Multiplicative-Additive Linear Logic (MALL) is faithfully interpretable in BCI with constants $\wedge$ and $\top$, and Cyclic MALL [30] is faithfully interpretable in AL0 with $\wedge$, $\top$ and an additional rule for a distinguished constant. For associative systems without exchange, the rules for $\wedge$ are:

$$(\wedge \mathrm{L1})\ \frac{\Gamma, A, \Delta \vdash C}{\Gamma, B \wedge A, \Delta \vdash C},\ (\wedge \mathrm{L2})\ \frac{\Gamma, A, \Delta \vdash C}{\Gamma, A \wedge B, \Delta \vdash C},$$

$$(\wedge \mathrm{R})\ \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \wedge B}.$$

For associative systems with exchange, Δ can be dropped in rules ($\wedge$L1) and ($\wedge$L2). The axiom for $\top$ is: $\Gamma \vdash \top$. The formulation of analogous rules for nonassociative systems is left to the reader.

Algebraic frames for Lambek logics are residuated semigroups, residuated monoids, residuated groupoids and others. *A residuated groupoid* is a structure $(M, \leq, \cdot, \backslash, /)$ such that $(M, \leq)$ is a poset, and $\cdot, \backslash, /$ are binary operations on M, satisfying the equivalences:

$$(\mathrm{RES})\ ab \leq c \text{ iff } b \leq a \to c \text{ iff } a \leq c \leftarrow b,$$

for all $a, b, c \in M$. As a consequence, $(M, \leq, \cdot)$ is a partially ordered groupoid (p.o. groupoid), that means, the following condition holds true:

$$(\mathrm{MON}) \text{ if } a \leq b \text{ then } ca \leq cb \text{ and } ac \leq bc.$$

A unit in M is an element 1 such that $1a = a = a1$, for all $a \in M$. A residuated groupoid such that $\cdot$ is an associative operation is called *a residuated semigroup*, and a residuated semigroup with unit is called *a residuated monoid*.

Let M be a residuated groupoid. A model over M is a pair (M, μ) such that μ is a mapping from the set of formulas into M which satisfies the following conditions:

$$(\mathrm{MOD})\ \mu(A \otimes B) = \mu(A)\mu(B),$$

$$\mu(A \to B) = \mu(A) \backslash \mu(B),\ \mu(B \leftarrow A) = \mu(B)/\mu(A),$$

for any formulas A, B. Then, μ is extended to be defined on formula structures, by setting $\mu(X \circ Y) = \mu(X)\mu(Y)$. If 1 is the unit in M, then we set $\mu(0) = 1$ and $\mu(1) = 1$. For associative systems, models are based on residuated semigroups M; then, μ can be defined on nonempty strings of formulas, by setting:

$$\mu(A_1 \ldots A_n) = \mu(A_1) \cdots \mu(A_n).$$

If 1 is the unit in M, we also set $\mu(\epsilon) = 1$. A sequent $\Gamma \vdash A$ (resp. $X \vdash A$) is said to be *true* in model (M, μ), if $\mu(\Gamma) \leq \mu(A)$ (resp. $\mu(X) \leq \mu(A)$). A sequent is said to be *valid* in a frame M if it is true in all models over M.

It can easily be shown that the sequents provable in NL are precisely those which are valid in all residuated groupoids. Thus, NL is complete with respect to residuated groupoids. The soundness is easy, and the completeness can be proven by the Lindenbaum model. Analogously, NL0 and NL1 are complete with respect to residuated groupoids with unit, AL is complete with respect to residuated semigroups, and AL0, AL1 are complete with respect to residuated monoids [3, 4].

An element $\top \in M$ is called *a top* of M if $a \leq \top$, for all $a \in M$. A binary operation $\wedge$ on M is called *a meet* if the following equivalence:

$$\text{(MEET) } a \leq b \wedge c \text{ iff } a \leq b \text{ and } a \leq c,$$

holds, for all $a, b, c \in M$. If M admits meet and/or $\top$, then one assumes:

$$\mu(A \wedge B) = \mu(A) \wedge \mu(B),\ \mu(\top) = \top,$$

provided that the corresponding operation symbols appear in the language. Now, each of the afore mentioned systems, enriched with $\wedge$ and/or $\top$, is complete with respect to the corresponding frames supplied with meet and/or top.

In this paper, we focus on special frames which consist of binary relations on a set. For $R, S \subseteq U^2$, one sets:

$$R \circ S = \{(x, y) \in U^2 : \exists z((x, z) \in R \& (z, y) \in S)\},$$

$$R \backslash S = \{(x, y) \in U^2 : \forall z((z, x) \in R \Rightarrow (z, y) \in S)\},$$

$$S/R = \{(x, y) \in U^2 : \forall z((y, z) \in R \Rightarrow (x, z) \in S)\},$$

$$I_U = \{(x, y) \in U^2 : x = y\}.$$

Then, $(P(U^2, \subseteq, \circ, \backslash, /, I_U)$ is a residuated monoid. It admits the meet operation $R \wedge S = R \cap S$ and the top $\top = U^2$. This frame will be called *the full relational frame* on U.

Let $T \subseteq U^2$. For $R, S \subseteq T$, one sets:

$$R \circ S = \{(x, y) \in T : \exists z((x, z) \in R \& (z, y) \in S)\},$$

$$R \backslash S = \{(x, y) \in T : \forall z((z, x) \in R \& (z, y) \in T \Rightarrow (z, y) \in S)\},$$

$$S/R = \{(x, y) \in T : \forall z((y, z) \in R \& (x, z) \in T \Rightarrow (x, z) \in S)\}.$$

Then, $(P(T), \subseteq, \circ, \backslash, /)$ is a residuated groupoid [28]. It admits the meet operation $R \wedge S = R \cap S$ and top $\top = T$. This frame will be called *the restricted relational frame* determined by T. If T is transitive, that means: $(x, y) \in T$ and $(y, z) \in T$ entail $(x, z) \in T$, then operation $\circ$ is the same as for the full relational frame on U, and one can omit the conjunct $(z, y) \in T$ (resp. $(x, z) \in T$) in the definition of $R \backslash S$ (resp. S/R).

In [1], it has been proven that AL0, AL1 are complete with respect to full relational frames, and AL is complete with respect to restricted relational frames determined by irreflexive and transitive relations. Actually, the authors prove some representation theorems: (i) every residuated monoid is embeddable into a full relational frame, (ii) every residuated semigroup is embeddable into a restricted relational frame determined by an irreflexive and transitive relation. [28] shows that every residuated groupoid is embeddable into a restricted relational frame, which yields the completeness of NL with respect to restricted relational frames.

In [9], similar results have been proven by applying labelled formulas in the sense of D. Gabbay and canonical models for arbitrary theories based on Lambek logics. Actually, (ii) has been strengthened: every residuated semigroup is embeddable into a restricted relational frame determined by an irreflexive linear ordering. Also: every residuated monoid satisfying: $1 \leq ab$ entails $1 \leq a$ and $1 \leq b$, is embeddable into a restricted relational frame determined by a reflexive linear ordering. This yields the completeness of AL0, AL1 with respect to restricted relational frames determined by a reflexive linear ordering.

As matter of fact, representation theorems yield the strong completeness of the respective systems: the sequents provable in the system from a set of new axioms are precisely those which are true in all models over the respective frames such all new axioms are true in the model. In [9], the way of reasoning was opposite: the strong completeness has been shown by means of a canonical model and used to prove the appropriate representation theorem.

In this paper, we present a uniform method of proof for representation theorems of that kind, for associative and nonassociative structures, with and without unit, top, meet. We use purely algebraic and set-theoretic tools, but no logical tools (like canonical models in [9]) and no graph-theoretic considerations (like transitivity checking in [1]).

2 A representation theorem

First, we give the proof of the following representation theorem: every residuated semigroup is embeddable into a restricted relational frame determined by an irreflexive linear ordering. Then, we show how to modify the proof for the case of other algebras.

Let $(M, \leq, \cdot, \backslash, /)$ be a residuated semigroup. By *a partial representation* of M we mean a triple (U, T, f) such that U is a nonempty set, $T \subseteq U^2$ is an irreflexive linear ordering, and f is a mapping from M into $P(T)$, satisfying the following conditions:

(PR1) $f(a) \circ f(b) \subseteq f(ab)$,

(PR2) if $a \leq b$ then $f(a) \subseteq f(b)$,

for all $a, b \in M$. A partial representation of M is called *a representation* of M if it satisfies the following conditions:

(R1) $f(a) \circ f(b) = f(ab)$,

(R2) $f(a\backslash b) = f(a)\backslash f(b)$,

(R3) $f(a/b) = f(a)/f(b)$,

(R4) $a \leq b$ iff $f(a) \subseteq f(b)$,

for all $a, b \in M$. Operations $\backslash$ and $/$ on relations are understood in the sense of restricted relational frames.

Lemma 1 *Let (U,T,f) be a partial representation of M. Then, $f(a\backslash b) \subseteq f(a)\backslash f(b)$ and $f(a/b) \subseteq f(a)/f(b)$, for all $a, b \in M$.*

PROOF. For $a, b \in M$, we have $a(a\backslash b) \leq b$, by $a\backslash b \leq a\backslash b$ and (RES). Then, $f(a) \circ f(a\backslash b) \subseteq f(b)$, by (PR1) and (PR2), hence $f(a\backslash b) \subseteq f(a)\backslash f(b)$, by (RES). The second inclusion can be shown in a dual way. Q.E.D.

Let $D = (U,T,f)$ and $D' = (U',T',f')$ be partial representations of M. D' is called *a conservative extension* of D if $U \subseteq U'$, $T = T' \cap U^2$ and $f(a) = f'(a) \cap T$, for all $a \in M$; we write $D \sqsubseteq D'$ if D' is a conservative extension of D.

Let T be an irreflexive linear ordering on a nonempty set U, and let f be a mapping from M into $P(T)$. By *the closure* of f we mean a mapping $c(f)$ from M into $P(T)$, defined as follows: $(x,y) \in c(f)(a)$ iff there exist $x_0, \ldots, x_n \in U$ and $a_1, \ldots, a_n \in M$, $n > 0$, such that $x = x_0$, $y = x_n$, $(x_{i-1}, x_i) \in f(a_i)$, for all $0 < i \leq n$, and $a_1 \cdots a_n \leq a$.

Lemma 2 *$(U,T,c(f))$ is a partial representation of M, and $f(d) \subseteq c(f)(d)$, for all $d \in M$.*

PROOF. (PR2) is obvious. We prove (PR1). Assume $(x,y) \in c(f)(a)$ and $(y,z) \in c(f)(b)$. There exist $x_0, \ldots, x_m$ and $a_1, \ldots, a_m$ such that $x = x_0$, $y = x_m$, $(x_{i-1}, x_i) \in f(a_i)$, for all $0 < i \leq m$, and $a_1 \cdots a_m \leq a$. There exist $y_0, \ldots, y_n$ and $b_1, \ldots, b_n$ such that $y = y_0$, $z = y_n$, $(y_{i-1}, y_i) \in f(b_i)$, for all $0 < i \leq n$, and $b_1 \cdots b_n \leq b$. By (MON), $a_1 \cdots a_m b_1 \cdots b_n \leq ab$. Then, the sequence $x_0, \ldots, x_m = y_0, \ldots, y_n$ witnesses $(x,z) \in c(f)(ab)$. Assume $(x,y) \in f(d)$. Then, x, y and d witness $(x,y) \in c(f)(d)$. Q.E.D.

We describe three constructions of a partial representation of M which is a conservative extension of a given partial representation of M. Similar ideas have been employed in [16, 9], but here they are expressed without any tools from deductive systems.

CONSTRUCTION 1. Let (U,T,f) be a partial representation of M and $(x',y') \in f(ab)$. Let $u \notin U$. We set $U' = U \cup \{u\}$. T' equals T on U and contains all pairs (z,u), for $z \in U$ such that $(z,x') \in T$ or $z = x'$, and all pairs (u,z), for $z \in U$ such that $(x',z) \in T$. So, T' is an irreflexive linear ordering on U' in which u is positioned immediately after x'. Finally, we define $f'(a) = f(a) \cup \{(x',u)\}$, $f'(b) = f(b) \cup \{(u,y')\}$, and $f'(c) = f(c)$, for all $c \in M$ such that $c \neq a$ and $c \neq b$. So, f' maps M into $P(T')$, and $f'(d) \cap T = f(d)$, for all $d \in M$.

Lemma 3 $(U,T,f) \sqsubseteq (U',T',c(f'))$ *and* $(x',u) \in c(f')(a)$, $(u,y') \in c(f')(b)$.

PROOF. By lemma 2, $(U',T',c(f'))$ is a partial representation of M. Clearly, $U \subseteq U'$ and $T = T' \cap U^2$. Also, for $d \in M$, $f(d) \subseteq f'(d)$ and $f'(d) \subseteq c(f')(d)$, which yields $f(d) \subseteq c(f')(d)$. Assume $(x,y) \in c(f')(d)$, where $(x,y) \in T$. Then, there exist $x_0, \ldots, x_n \in U'$ and $a_1, \ldots, a_n \in M$, $n > 0$, such that $x = x_0$, $y = x_n$, $(x_{i-1}, x_i) \in f'(a_i)$, for all $0 < i \leq n$, and $a_1 \cdots a_n \leq d$. Let k be the number of occurrences of u among $x_0, \ldots, x_n$. By induction on k, we prove $(x,y) \in f(d)$. (Actually, since $(u,u) \notin T$, then $k \leq 1$, but the inductive argument will be useful in variants of this proof, discussed later on.) Let $k=0$. Then, (x,y) belongs to $f(a_1)\circ\cdots\circ f(a_n)$ which is contained in $f(d)$, by (PR1), (PR2). Assume the thesis holds for k. We prove it for $k+1$. Let i be the least integer such that $x_i=u$. Clearly, $0<i<n$. We have $(x_{i-1},u)\in f'(a_i)$ and $(u,x_{i+1})\in f'(a_{i+1})$. Then, $a_i=a$, $a_{i+1}=b$, and $x_{i-1}=x'$, $x_{i+1}=y'$. Since $(x',y')\in f(ab)$, then $(x,y)\in c(f')(d)$ can be witnessed by $x_0,\ldots,x_{i-2},x',y',x_{i+2},\ldots,x_n$ and $a_1\cdots a_{i-1}aba_{i+2}\cdots a_n\leq d$, with u occurring k times in the former sequence. Consequently, $(x,y)\in f(d)$, by the induction hypothesis. We have shown $c(f')(d)\cap T=f(d)$. Since $(x',u)\in f'(a)$. $(u,y')\in f'(b)$. then the second part of the thesis is also true. Q.E.D.

Construction 2. Let (U,T,f) be a partial representation of M, $(x'y')\in T$ and $(x',y')\notin f(a\backslash b)$. Let $u\notin U$. We set $U'=U\cup\{u\}$. T' equals T on U and contains all pairs (u,z), for $z\in U$. So, T' is an irreflexive linear ordering on U' in which u is the first element. Finally, we define $f'(a)=f(a)\cap\{(u,x')\}$ and $f'(c)=f(c)$, for all $c\neq a$. Then, f' maps M into $P(T')$, and $f'(d)\cap T=f(d)$, for all $d\in M$.

Lemma 4 $(U,T,f)\sqsubseteq(U',T',c(f'))$ *and* $(u,x')\in c(f')(a)$, $(u,y')\notin c(f')(b)$.

Proof. Again, $(U',T',c(f'))$ is a partial representation of M, $U\subseteq U'$, $T'\cap U^2=T$ and $f(d)\subseteq c(f')(d)$, for all $d\in M$. Assume $(x,y)\in c(f')(d)$, $(x,y)\in T$. There exist $x_0,\ldots,x_n\in U'$ and $a_1,\ldots,a_n\in M$ which satisfy the same conditions as in the proof of lemma 3. Since no pair of the form (z,u) belongs to $f'(a_i)$, for $0<i\leq n$, then the number k must be 0, and we argue as above, for the case $k=0$. Consequently, $c(f')(d)\cap T=f(d)$. Clearly, $(u,x')\in c(f')(a)$. We show $(u,y')\notin c(f')(b)$. Suppose the contrary. There exist $x_0,\ldots,x_n$ and $a_1,\ldots,a_n$ such that $x_0=u$, $x_n=y'$, and the remainder is the same as above (for $d=b$). By the same reason as above, x_0 is the only occurrence of u among $x_0,\ldots,x_n$. We have $(u,x_1)\in f'(a_1)$, hence $x_1=x'$, $a_1=a$. Since $x_n=y'\neq x'$, then $n>1$. Further, $aa_2\cdots a_n\leq b$, which yields $a_2\cdots a_n\leq a\backslash b$, by (RES), and consequently, $(x',y')\in c(f')(a\backslash b)$. But $(x',y')\in T$, hence $(x',y')\in f(a\backslash b)$, by the first part of the lemma. This contradicts the assumption of construction 2. Q.E.D.

Construction 3. This construction is dual to construction 2. Let (U,T,f) be a partial representation of M, $(x',y')\in T$ and $(x',y')\notin f(a/b)$. Let $u\notin U$. We set $U'=U\cup\{u\}$. T' equals T on U and contains all pairs (z,u), for $z\in U$. So, T' is an irreflexive linear ordering on U' in which u is the last element. Finally, we define $f'(b)=f(b)\cup\{(y',u)\}$ and $f'(c)=f(c)$, for all $c\neq b$. Then, f' maps M into $P(T')$, and $f'(d)\cap T=f(d)$, for all $d\in M$. The following lemma is dual to lemma 4.

Lemma 5 $(U,T,f)\sqsubseteq(U',T',c(f'))$ *and* $(y',u)\in c(f')(b)$, $(x',u)\notin c(f')(a)$.

Let $D=(U,T,f)$, $D'=(U',T',f')$ be partial representations of M. We say that D' *witnesses* D, if $D\sqsubseteq D'$ and, for all $(x,y)\in T$ and $a,b\in M$, the following conditions hold true:

(W1) if $(x,y) \in f(ab)$, then $(x,y) \in f'(a) \circ f'(b)$,

(W2) if $(x,y) \notin f(a\backslash b)$, then $(x,y) \notin f'(a)\backslash f'(b)$,

(W3) if $(x,y) \notin f(a/b)$, then $(x,y) \notin f'(a)/f'(b)$.

Let $(D_\alpha)_{\alpha<\gamma}$, $\gamma > 0$, be a (possibly transfinite) sequence of partial representations of M such that $D_\alpha \sqsubseteq D_\beta$, for all $\alpha < \beta$. We define the limit $D = (U,T,f)$, by setting: $U = \bigcup_{\alpha<\gamma} U_\alpha$, $T = \bigcup_{\alpha<\gamma} T_\alpha$, and $f(a) = \bigcup_{\alpha<\gamma} f_\alpha(a)$, for all $a \in M$. The proof of lemma 6 is easy.

Lemma 6 *$D_\alpha \sqsubseteq D$, for all $\alpha < \gamma$.*

Lemma 7 *Let $(D_n)_{n<\omega}$ be an $\omega-$sequence of partial representations of M such that D_{n+1} witnesses D_n, for all $n < \omega$, and D_0 satisfies (R4). Let $D = (U,T,f)$ be the limit of this sequence. Then, D is a representation of M.*

Proof. By lemma 6, $D_n \sqsubseteq D$, for all $n < \omega$, hence D is a partial representation of M. We denote $D_n = (U_n, T_n, f_n)$.

We show that D satisfies (R4). D satisfies (PR2). Assume $a \not\leq b$. Then, $f_0(a) \not\subseteq f_0(b)$, since D_0 satisfies (R4). So, there exists $(x,y) \in f_0(a)$, $(x,y) \notin f_0(b)$. Since $D_0 \sqsubseteq D$, then $(x,y) \in f(a)$, $(x,y) \notin f(b)$, which yields $f(a) \not\subseteq f(b)$.

We show that D satisfies (R1). D satisfies (PR1). Assume $(x,y) \in f(ab)$. Then, there exists $n < \omega$ such that $(x,y) \in f_n(ab)$. Since D_{n+1} witnesses D_n, then $(x,y) \in f_{n+1}(a) \circ f_{n+1}(b)$, which yields $(x,y) \in f(a) \circ f(b)$.

We show that D satisfies (R2). By lemma 1, $f(a\backslash b) \subseteq f(a)\backslash f(b)$. Assume $(x,y) \in T$, $(x,y) \notin f(a\backslash b)$. Then, there exists $n < \omega$ such that $(x,y) \in T_n$, $(x,y) \notin f_n(a\backslash b)$. Since D_{n+1} witnesses D_n, then $(x,y) \notin f_{n+1}(a)\backslash f_{n+1}(b)$. So, there exists $z \in U_{n+1}$ such that $(u,x) \in f_{n+1}(a)$, $(u,y) \notin f_{n+1}(b)$. Since $D_{n+1} \sqsubseteq D$, then $(u,x) \in f(a)$, $(x,y) \notin f(b)$, hence $(x,y) \notin f(a)\backslash f(b)$. In a dual way one shows that D satisfies (R3). Q.E.D.

Lemma 8 *Let $D = (U,T,f)$ be a partial representation of M. Then, there exists D' such that D' witnesses D.*

Proof. The proof uses the axiom of choice. Let κ be the maximum of the cardinalities of U and M. First, assume that κ is infinite. The set $U^2 \times M^2 \times \{1,2,3\}$ is of cardinality κ. Let $(t_\alpha)_{\alpha<\kappa}$ be a sequence of all elements of this set. We shall define a chain $(D_\alpha)_{\alpha<\kappa}$ of partial representations of M such that $D_\alpha \sqsubseteq D_\beta$ whenever $\alpha < \beta$. We denote $D_\alpha = (U_\alpha, T_\alpha, f_\alpha)$. Put $D_0 = D$. Assume D_α, for $\alpha < \gamma$, have already been defined. If γ is a limit ordinal, we set D_γ to be the limit of the chain $(D_\alpha)_{\alpha<\gamma}$. By lemma 6, $D_\alpha \sqsubseteq D_\gamma$, for all $\alpha < \gamma$.

Let $\gamma = \beta+1$. Take $t_\beta = (x', y', a, b, i)$. We consider four cases. (I) $i = 1$ and $(x', y') \in f_\beta(ab)$. We define $D_{\beta+1}$ by construction 1 with $U = U_\beta$, $T = T_\beta$, $f = f_\beta$, and $D_{\beta+1} = (U', T', c(f'))$. (II) $i = 2$ and $(x', y') \in T_\beta$, $(x', y') \notin f_\beta(a\backslash b)$. We define $D_{\beta+1}$ by construction 2. (III) $i = 3$ and $(x', y') \in T_\beta$, $(x, y) \notin f_\beta(a/b)$. We define $D_{\beta+1}$ by construction 3. (IV) cases (I), (II), (III) do not hold. We define $D_{\beta+1} = D_\beta$. By lemmas 3, 4, 5, $D_\beta \sqsubseteq D_{\beta+1}$, hence $D_\alpha \sqsubseteq D_\gamma$, for all $\alpha < \gamma$.

Let $D' = (U', T', g)$ be the limit of $(D_\alpha)_{\alpha<\kappa}$. By lemma 6, $D_\alpha \sqsubseteq D'$, for all $\alpha < \kappa$.

We show (W1). Assume $(x', y') \in T$, $(x', y') \in f(ab)$. Then, there exists $\beta < \kappa$ such that $t_\beta = (x', y', a, b.1)$. Case (I) holds and, by lemma 3, $(x', y') \in f_{\beta+1}(a) \circ f_{\beta+1}(b)$, which yields $(x', y') \in g(a) \circ g(b)$.

We show (W2). Assume $(x', y') \in T$, $(x', y') \notin f(a\backslash b)$. Then, there exists $\beta < \kappa$ such that $t_\beta = (x', y', a, b, 2)$. Since $D \sqsubseteq D_\beta$, then $(x', y') \in T_\beta$, $(x', y') \notin f_\beta(a\backslash b)$. So, case (II) holds and, by lemma 4, there is u such that $(u, x') \in f_{\beta+1}(a)$, $(u, y') \notin f_{\beta+1}(b)$. Since $D_{\beta+1} \sqsubseteq D'$, then $(u, x') \in g(a)$, $(u, y') \notin g(b)$, and consequently, $(x', y') \notin g(a)\backslash g(b)$. The proof of (W2) is dual.

If U and M are finite, then $U^2 \times M^2 \times \{1, 2, 3\}$ is also finite, and we consider finite sequences $(t_j)_{j<n}$ and $(D_j)_{j\leq n}$. Now, $D' = D_n$. Then, the proof goes as above. Q.E.D.

We are ready to prove:

Theorem 1 *For any residuated semigroup M, there exists a representation of M.*

Proof. Let M be a residuated semigroup. First, we define a partial representation of M which satisfies (R4). Here, we present one method. For every $a \in M$, choose different elements x_a, y_a. Let U consist of all x_a, y_a, for $a \in M$. Let T be an arbitrary irreflexive linear ordering on U which contains all pairs (x_a, y_a) (we use the axiom of choice). For $a \in M$, we define:

$$f(a) = \{(x_c, y_c) : c \leq a\}.$$

Take $D = (U, T, f)$. (PR2) is obvious. If $a \not\leq b$, then $(x_a, y_a) \in f(a)$, $(x_a, y_a) \notin f(b)$, hence (R4) is true. (PR1) also holds, since $f(a) \circ f(b) = \emptyset$, for all $a, b \in M$.

We form a chain $(D_n)_{n<\omega}$ of partial representations of M such that $D_0 = D$, and D_{n+1} witnesses D_n, for all $n < \omega$; the existence of such a chain follows from lemma 8. By lemma 7, the limit of this chain is a representation of M. Q.E.D.

Notice that, for a countable M, all D_n's are countable, hence in the resulting representation (U, T, f) the universe U is countable. Therefore, every countable residuated semigroup is embeddable into the restricted relational frame determined by a countable, irreflexive linear ordering T.

3 Variants

We shall briefly discuss different modifications of the above proof appropriate for other representation theorems.

Irreflexive partial orderings. In [1] it has been shown that every residuated semigroup is embeddable into the restricted relational frame determined by an irreflexive partial ordering. The proof uses the axiom of choice. By applying the above methods, the axiom of choice can be avoided. We redefine the notions of a partial representation and a representation, assuming T to be an irreflexive partial ordering on U. Lemma 8 can be proven without the axiom of choice. For every quadruple (x', y', a, b) such that $(x', y') \in f(ab)$, one introduces a new element u. For every quadruple (x', y', a, b) such that $(x', y') \notin f(a\backslash b)$, one introduces a new element v. For every quadruple (x', y', b, a) such that $(x', y') \notin f(b/a)$, one introduces a new element w. U' is U enriched with all new elements. T' is the smallest irreflexive partial ordering on U' which contains T and all pairs $(x', u), (u, y')$, for quadruples of the first kind and the corresponding new elements u, in which all new elements v are minimal elements and all new elements w are maximal elements. $f'(a)$ is $f(a)$ enriched with all pairs (x', u) such that there exists a quadruple (x', y', a, b) of the first kind, and u is the corresponding new element, all pairs (u, y') such that there exists a quadruple (x', y', b, a) of the first kind, and u is the corresponding new element, all pairs (v, x') such that there exists a quadruple (x', y', a, b) of the second kind, and v is the corresponding new element, and all pairs (y', w) such that there exists a quadruple (x', y', b, a) of the third kind, and w is the corresponding new element. We define $D' = (U', T', c(f'))$. The proof of $D \sqsubseteq D'$ joins arguments used in the proofs of lemmas 3, 4, 5, and similarly for (W1), (W2), (W3). Finally, in the proof of theorem 1 D_0 can be formed out without the axiom of choice: T_0 consists of all pairs (x_a, y_a), for $a \in M$.

Residuated groupoids. It has been shown in [28] that every residuated groupoid is embeddable into a restricted relational frame (now, T need not be transitive). The proof uses logical tools. Here, we show how to prove this theorem by the above methods. In the definitions of a partial representation and a representation, we assume T to be an arbitrary relation on U.

Since the product is nonassociative, $a_1 \cdots a_n \leq a$ is not meaningful, unless we fix a bracketing on $a_1 \cdots a_n$. Therefore, we say: $a_1 \cdots a_n \leq a$ with a bracketing p. If $a_1 \cdots a_n$ is bracketed by p, then the notion of *a constituent* of $a_1 \cdots a_n$ with respect to p is defined in a natural way. For example, the constituents of $((ab)(cd))$ are: $abcd$, ab, cd, a, b, c, d.

Let $(M, \leq, \cdot, \backslash, /)$ be a residuated groupoid, and let $T \subseteq U^2$. Let p be a bracketing on $a_1 \cdots a_n$. We say that a sequence $x_0, \ldots, x_n$ is *compatible* with p, T if, for every constituent $a_i \cdots a_j$ of $a_1 \cdots a_n$ with respect to p, it holds $(x_{i-1}, x_j) \in T$.

Let f be a mapping from M into $P(T)$. We define *the closure* of f as the mapping $c(f)$ from M into $P(T)$, satisfying: $(x, y) \in c(f)(a)$ iff there exist $x_0, \ldots, x_n \in U$ and $a_1, \ldots, a_n \in M$, $n > 0$, such that $a_1 \cdots a_n \leq a$ with a bracketing p, the sequence $x_0, \ldots, x_n$ is compatible with p, T, $x_0 = x$, $x_n = y$, and $(x_{i-1}, x_i) \in f(a_i)$, for all $0 < i \leq n$.

Lemma 1 is true. The proof of lemma 2 can easily be modified. If p is the given bracketing on $a_1 \cdots a_m$, and q on $b_1 \cdots b_n$, then r on $a_1 \cdots a_m b_1 \cdots b_n$ is $((p)(q))$. Now, $a_1 \cdots a_m b_1 \cdots b_n \leq ab$ with bracketing r, and the sequence $x_0, \ldots, x_m = y_0, \ldots, y_n$ is compatible with r, T (notice that $(x, y) \in T$, by the definition of $\circ$ in restricted relational frames).

We assume that T is irreflexive in all partial representations (U, T, f) to be considered. In construction 1 we take $T' = T \cup \{(x', u), (u, y')\}$ and the remainder as above. In the proof of lemma 3, if u occurs among $x_0, \ldots, x_n$, it must be $u = x_i$, $0 < i < n$, $x_{i-1} = x'$, $x_{i+1} = y'$, and $a_i a_{i+1} = ab$ is a constituent of $a_1 \cdots a_n$. Assume it is not the case. Then, $a_i a_{i+1}$ is not a constituent of $a_1 \cdots a_n$. There exist k, l such that $a_k \cdots a_i$ and $a_{i+1} \cdots a_l$ are constituents of $a_1 \cdots a_n$, and either $k < i$, or $i + 1 < l$. Consider the first case. Let k be the greatest integer less than i such that $a_k \cdots a_i$ is a constituent. Then, the bracketing is $((a_k \cdots a_{i-1})a_i)$. By compatibility, $(x_{k-1}, u) \in T'$, hence $x_{k-1} = x'$. Since $x_{i-1} = x'$, we get $(x', x') \in T'$, which is impossible. In the second case, the reasoning is similar. Then, we can continue as in the proof of lemma 3.

In construction 2 we take $T' = T \cup \{(u, x'), (u, y')\}$ and the remainder as above. The proof of lemma 4 uses arguments similar to those from the preceding paragraph. In construction 3 we take $T' = T \cup \{(x', u), (y', u)\}$ and the remainder as above. The proof of lemma 5 is dual. Proofs of lemmas 6, 7, 8 can be copied. In the proof of theorem 1, T_0 consists of all pairs (x_a, y_a), for $a \in M$. Again, one can avoid the axiom of choice, by changing the proof of lemma 8.

RESIDUATED MONOIDS. For $R, S \subseteq I_U$, we have $R \circ S = R \cap S$. In residuated monoids, $a \leq 1$ and $b \leq 1$ entails $ab \leq a$ and $ab \leq b$, hence $ab \leq a \wedge b$ (if $a \wedge b$ exists), but not necessarily $ab = a \wedge b$. Then, an embedding of a residuated monoid M into the full relational frame $P(U^2)$ is not required to satisfy $f(1) = I_U$. One stipulates a weaker condition:

$$(1)\ 1 \leq a \text{ iff } I_U \subseteq f(a),$$

for all $a \in M$. This yields the completeness theorem for AL0, AL1 in the usual sense: A is provable iff $1 \leq [A]$ in the Lindenbaum algebra iff $I_U \subseteq f([A])$ in the relational frame iff A is true the relational model.

In [1], it has been shown that every residuated monoid is embeddable into the full relational frame on some universe U. Again, this result can be proven by the above methods, and the axiom of choice can be eliminated. Now, we take $T = U^2$ in all partial representations (so, T can simply be dropped). Further, a partial representation of M must satisfy:

(PR3) if $1 \leq a$ then $I_U \subseteq f(a)$,

for all $a \in M$, and a representation of M must satisfy (1).

Lemma 1 is true. The closure of f is defined as above, with admitting $n = 0$ if $1 \leq a$. This yields $I_U \subseteq c(f)(a)$ if $1 \leq a$. Now, lemma 2 can be proven as above. In constructions 1, 2, 3 we drop manipulations with T and preserve the remainder. Proofs of lemmas 3, 4, 5 can essentially be copied, with adding case $n = 0$ which is easy. An additional construction is needed.

CONSTRUCTION 4. Let $D = (U, f)$ be a partial representation of M and $1 \not\leq a$. Let $u \notin U$. We set $U' = U \cup \{u\}$ and consider $D' = (U', c(f))$. We prove:

Lemma 9 *$D \sqsubseteq D'$ and $I_{U'} \not\subseteq c(f)(a)$.*

PROOF. By lemma 2, D' is a partial representation of M. Assume $(x, y) \in c(f)(d)$, for $x, y \in U$. There exist $x_0, \ldots, x_n$ and $a_1, \ldots, a_n$ such that $x_0 = x$, $x_n = y$, $a_1 \ldots a_n \leq d$, and $(x_{i-1}, x_i) \in f(a_i)$, for $0 < i \leq n$. Assume $n > 0$. Since no pair from $f(a_i)$ contains u, we can proceed as in the proof of lemma 3 for $k = 0$. Assume $n = 0$. Then $1 \leq d$ and $x = y$. By (PR3) for D, $(x, y) \in f(d)$. Clearly, $(u, u) \notin c(f)(a)$, since $1 \not\leq a$. Q.E.D.

The proof of lemma 6 can be copied. In the definition of witnessing, we add the condition:

(W4) if $1 \not\leq a$ then $I_{U'} \not\subseteq f'(a)$,

for all $a \in M$. Then, lemma 7 can easily be proven. In the proof of lemma 8, the sequence $(t_\alpha)_{\alpha<\kappa}$ contains all elements of the set $(U^2 \times M^2) \cup M'$, where $M' = \{a \in M : 1 \not\leq a\}$. If $t_\beta = a$, $a \in M'$, then $D_{\beta+1}$ is formed out from D_β by construction 4. By conservativity, D' witnesses D. In the proof of theorem 1, we can define U and f as above and take $D_0 = (U, c(f))$ (one might also set $D_0 = (U, c(g))$, where $g(a) = \emptyset$, for all $a \in M$). The axiom of choice can be eliminated by changing the proof of lemma 8.

REFLEXIVE LINEAR ORDERINGS. Every residuated monoid in which $1 \leq ab$ holds only if $1 \leq a$ and $1 \leq b$ is embeddable into the restricted relational frame determined by a reflexive linear ordering T [9]. This can be shown by our methods, joining elements of the proofs for residuated semigroups and residuated monoids. We omit details.

RESIDUATED GROUPOIDS WITH UNIT. Every residuated groupoid with unit is embeddable into the restricted relational frame determined by a reflexive relation T. This is shown by mixing the proofs for residuated groupoids and residuated monoids. Now, partial representations and representations are defined as for residuated groupoids, with T reflexive. Additionally, representations must satisfy (1), and partial representations must satisfy (PR1)-(PR3) and:

(PR4) if $1 \not\leq a$ then $f(a)$ is irreflexive,

for all $a \in M$. Now, we can proceed as in the proof for residuated groupoids; in the definition of $c(f)$ we add the possibility $n = 0$ if $1 \leq a$. In lemma 2, we must assume that f satisfies (PR4). In proofs of lemmas 3, 4, 5, we need some changes. Look at the proof that $D \sqsubseteq D'$, sketched in the fragment concerning residuated groupoids. We have rejected the possibility $(x', x') \in T'$, since T' is irreflexive. Now, $(x', x') \in T'$ is true, but we infer $(x', x') \in f(a_k \cdots a_{i-1})$ with the induced bracketing on $a_k \cdots a_{k-1}$, by (PR1), (PR2), and consequently, $1 \leq a_k \cdots a_{i-1}$, by (PR4). Therefore, $x_0, \ldots, x_{k-1}, x_i, \ldots, x_n$ and $a_1 \cdots a_{k-1} a_i \cdots a_n$ are shorter sequences, proving $(x, y) \in c(f')(d)$. Accordingly, arguments in lemmas 3, 4, 5 can be saved, if one always takes the least possible n. Clearly, (PR4) is preserved by these constructions. The remainder of the proof is more or less the same as for residuated groupoids. Construction 4 is not needed.

RESIDUATED SEMIGROUPS WITH MEET. Assume that M admits meet. We stipulate that every partial representation (U, T, f) of M satisfies:

$$f(a \wedge b) = f(a) \cap f(b), \text{ for all } a, b \in M.$$

Lemma 1 is true. We preserve the definition of closure. Lemma 2 is true except for the claim that $c(f)$ satisfies the above equality. Lemmas 3, 4, 5 remain true. Let us consider lemma 3. We assume that $D = (U, T, f)$ is a partial representation of M, and we prove that $D' = (U', T', c(f'))$ is a partial representation of M. We must show:

$$c(f')(d \wedge e) = c(f')(d) \cap c(f')(e), \text{ for all } d, e \in M.$$

The inclusion $\subseteq$ is easy. We prove the converse inclusion. Let $(x, y) \in c(f')(d) \cap c(f')(e)$. Let $x \neq u$ and $y \neq u$. By conservativity, $(x, y) \in f(d) \cap f(e)$, which yields $(x, y) \in f(d \wedge e)$, hence $(x, y) \in c(f')(d \wedge e)$. Assume $x = u$. Then, $y \neq u$. There exist $x_0, \ldots, x_m$ and $a_1, \ldots, a_m$ such that $u = x_0$, $y = x_m$, $(x_{i-1}, x_i) \in f'(a_i)$, for all $0 < i \leq m$, and $a_1 \cdots a_m \leq d$. There exist $y_0, \ldots, y_n$ and $b_1, \ldots, b_n$ such that $u = y_0$, $y = y_n$, $(y_{i-1}, y_i) \in f'(b_i)$, for all $0 < i \leq n$, and $b_1 \cdots b_n \leq e$. Assume $m = 1$. Then, $y = x_1 = y'$, by construction 1, hence also $n = 1$ (otherwise, $(y', y') \in T'$, which is impossible). We have $a_1 = b_1 = b$, by construction 1, and $b \leq d$, $b \leq e$, which yields $b \leq d \wedge e$. Consequently, $(x, y) \in c(f')(d \wedge e)$. Assume $n = 1$. Then, $m = 1$, and we argue as above. Assume $m > 1$ and $n > 1$. We have $x_1 = y_1 = y'$ and $a_1 = b_1 = b$, as above. By (RES), $a_2 \cdots a_m \leq b \backslash d$ and $b_2 \cdots b_m \leq b \backslash e$. Thus, $(y', y) \in c(f')(b \backslash d)$ and $(y', y) \in c(f')(b \backslash e)$. By conservativity, $(y', y) \in f(b \backslash d)$ and $(y', y) \in f(b \backslash e)$, which yields $(y', y) \in f((b \backslash d) \wedge (b \backslash e))$. Now, $(b \backslash d) \wedge (b \backslash e) = b \backslash (d \wedge e)$ is true in residuated semigroups, hence $(y', y) \in f(b \backslash (d \wedge e))$. Consequently, $(x, y) \in c(f')(d \wedge e)$. For $y = u$, the argument is dual.

The partial representation D_0 in the proof of theorem 1 satisfies the above equality: $(x_c, y_c) \in f(a \wedge b)$ iff $c \leq a \wedge b$ iff $c \leq a$ and $c \leq b$ iff $(x_c, y_c) \in f(a) \cap f(b)$. It is easy to show that the limit of a chain of partial representations of M, satisfying the above equality, also satisfies this equality. Therefore, we can prove that every residuated semigroup admitting meet is embeddable into the restricted relational frame determined by an irreflexive linear ordering T, and the embedding f satisfies the equality (this theorem has been proven in [1] in a different way). Analogous results can be obtained for all other kinds of structures, considered in this paper.

STRUCTURES WITH TOP. We cannot expect $f(\top) = T$ or $f(\top) = U^2$ to be satisfied by all embeddings. Clearly, $f(\top)$ must contain all relations $f(a)$, for $a \in M$. We can define the restricted relational frame determined by $f(\top)$. Consider the case of residuated monoids. Since $\top\top \leq \top$ is true, then $f(\top)$ is a transitive relation. Then, for relations $f(a)$, $a \in M$, operations $\circ, \backslash, /$ in the sense of the full relational frame are the same as the corresponding operations in the sense of the restricted relational frame. Therefore, f can be treated as an embedding of M into the restricted relational frame which preserves top, and similarly for other kinds of structures.

MALL AND CYCLIC MALL. MALL is a propositional fragment of Classical Linear Logic [14]. Its formulas are formed out of propositional variables $p, q, r, \ldots$, negated propositional variables $p^\perp, q^\perp, r^\perp, \ldots$ and constants $0, 1, \perp, \top$ by means of binary operation symbols $\otimes, \oplus, \wedge, \vee$. (We use a notation following Troelstra.) Models are *phase spaces*, defined as follows. Let $(M, \cdot, 1)$ be a commutative monoid, and let $\perp^\star \subseteq M$. For $X, Y \subseteq M$, one defines:

$$X \circ Y = \{ab : a \in X \& b \in Y\},\ X \to Y = \{a : \forall b (b \in X \Rightarrow ba \in Y)\}.$$

Then, $(P(M), \subseteq, \circ, \to, \{1\})$ is a commutative residuated monoid. We have $X \backslash Y = X \to Y$. One defines $X^\perp = X \to \perp^\star$. A set $X \subseteq M$ is called *a fact* if $X = X^{\perp\perp}$ (equivalently, $X = Y^\perp$, for some $Y \subseteq M$). A phase space is denoted $(M, \perp^\star)$.

A model over $(M, \perp^\star)$ is given by an assignment μ which assigns facts to propositional variables. It is defined for all formulas, by setting:

$$\mu(p^\perp) = \mu(p)^\perp,$$

$$\mu(1) = (\perp^\star)^\perp,\ \mu(0) = \mu(1)^\perp,\ \mu(\top) = M,\ \mu(\perp) = M^\perp,$$

$$\mu(A \otimes B) = (\mu(A) \circ \mu(B))^{\perp\perp},\ \mu(A \oplus B) = (\mu(A)^\perp \circ \mu(B)^\perp)^\perp,$$

$$\mu(A \wedge B) = \mu(A) \cap \mu(B),\ \mu(A \vee B) = (\mu(A)^\perp \cap \mu(B)^\perp)^\perp.$$

A is said to be true in the model if $1 \in \mu(A)$. As shown in [14], theorems of MALL are formulas true in all models over phase spaces.

The representation theorems, established above, show that commutative residuated monoids $(P(M), \subseteq, \circ, \to, \{1\})$ are embeddable into full relational frames, and the embeddings preserve meet and top. This yields a relational semantics for MALL. In this semantics, formulas are interpreted as binary relations, and linear implication $A \to B = A^\perp \oplus B$ is interpreted as residuation: $\mu(A \to B) = \mu(A) \backslash \mu(B)$. It is easy to see that all logical constants of MALL are definable in terms of relational residuation, top and meet plus the special constant $\perp^\star$.

Models of Cyclic MALL [30] are based on cyclic phase spaces. Now, $(M, \cdot, 1)$ is a monoid, and $\perp^\star \subseteq M$ satisfies: $ab \in \perp^\star$ entails $ba \in \perp^\star$. The remainder is as above. The embeddings described above yield a semantics for Cyclic MALL, based on full relational frames.

At the end, we describe another, much simpler construction, working for right residuated monoids M, satisfying:

$$(2)\ a \backslash (bc) = (a \backslash b)c, \text{ for all } a, b, c \in M.$$

We define:

$$(3)\ f(a) = \{(x, y) \in M^2 : x \leq ay\}.$$

Assuming (2), we show that (M, f) is a representation of M. We prove (R1). Let $(x, y) \in f(a) \circ f(b)$. Then, $x \leq az$ and $z \leq by$, for some z, which yields $x \leq aby$, hence $(x, y) \in f(ab)$. Let $(x, y) \in f(ab)$. Then, $x \leq aby$, hence, for $z = by$, we have $(x, z) \in f(a)$ and $(z, y) \in f(b)$. We prove (R4). Clearly, $a \leq b$ entails $f(a) \subseteq f(b)$. Assume $a \not\leq b$. Then, $(a, 1) \in f(a)$, $(a, 1) \notin f(b)$, hence $f(a) \not\subseteq f(b)$. We prove (R2). The inclusion $f(a \backslash b) \subseteq f(a) \backslash f(b)$ holds, by the same argument as in the proof of lemma 1. We prove the converse inclusion. Let $(x, y) \in f(a) \backslash f(b)$. Then, for any z, if $(z, x) \in f(a)$, then $(z, y) \in f(b)$. We have $(ax, x) \in f(a)$, so $ax \leq by$, which yields $x \leq a\backslash(by)$. By (2), $x \leq (a\backslash b)y$, hence $(x, y) \in f(a\backslash b)$.

We have $f(1) = \{(x, y) : x \leq y\}$. So, $I_M \subseteq f(1)$. Clearly, if $(x, y) \in f(a)$ and $x' \leq x$, $y \leq y'$, then $(x', y') \in f(a)$. Therefore, if $I_M \subseteq f(a)$, then $f(1) \subseteq f(a)$, which yields $1 \leq a$, by (R4). This shows (1).

Conversely, if (M, f) is a representation of M, where f is defined by (3), then M satisfies (2). Every right residuated monoid (even semigroup) satisfies $(a\backslash b)c \leq a\backslash(bc)$. Assume that f, defined by (3), is an embedding. We have: $(x, y) \in f(a)\backslash f(b)$ iff $z \leq ax$ entails $z \leq by$, for all z, iff $ax \leq by$ iff $x \leq a\backslash(by)$. Since $f(a)\backslash f(b) = f(a\backslash b)$, then $x \leq a\backslash(by)$ iff $x \leq (a\backslash b)y$, for all $x, y \in M$, which yields $a\backslash(by) = (a\backslash b)y$. We have shown that M satisfies (2), and we have proven:

Theorem 2 *For any right residuated monoid M, the mapping f defined by (3) is an embedding of M into the relational monoid $P(M^2)$ if, and only if, M satisfies (2).*

A similar result can be obtained for left residuated monoids. We notice that right (resp. left) residuated monoids satisfying (2) (resp. the dual of (2)) are precisely right (resp. left) pregroups in the sense of [8], structures closely related to pregroups in the sense of [19]. One may define $a^r = a\backslash 1$ (resp. $a^l = 1/a$). Then, $aa^r \leq 1$ and

$$1 \leq a\backslash a = a\backslash(1a) = a^r a,$$

and similarly for the dual operation. Conversely, in a right (resp. left) pregroup, one defines $a\backslash b = a^r b$ (resp. $a/b = ab^l$), and the resulting structure is a right (resp. left) residuated monoid, satisfying (2) (resp. its dual). Several results on relational and functional representations of (right, left) pregroups has been proven in [7, 8].

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