Finite Restricted Powerset Models for the Lambek Calculus and Its Extensions

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Abstract

Lafont [11] gives an elegant proof of finite model property (fmp) for Multiplicative-Additive Linear Logic (MALL) and suggests it can be adapted to noncommutative linear logics of Abrusci [1]. In [4, 6, 9], fmp for certain fragments of MALL (e.g. BCI, BCI[$\wedge$]) and weaker systems (e.g. the product-free Lambek calculus) has been proven by a different method; whereas Lafont uses phase space models and quotient-structures, I employ cofinite powerset models and so-called barriers. In this paper I present proofs of fmp by the method of barriers; I simplify and generalize earlier approaches and use finite restricted powerset models instead of cofinite powerset models. The former seem to be more intuitive than the latter as partial models of natural language in the sense of Categorial Grammar [7, 8].

1 Basic notions

The Lambek calculus, introduced by Lambek [12], admits the following axioms and rules:

$$\text{(Id)}\ A \vdash A,$$

$$(\mathrm{L}{\to})\ \frac{\Gamma, B, \Gamma' \vdash C;\ \Delta \vdash A}{\Gamma, \Delta, A \to B, \Gamma' \vdash C},\quad (\mathrm{R}{\to})\ \frac{A, \Gamma \vdash B}{\Gamma \vdash A \to B},$$

$$(\mathrm{L}{\leftarrow})\ \frac{\Gamma, B, \Gamma' \vdash C;\ \Delta \vdash A}{\Gamma, B \leftarrow A, \Delta, \Gamma' \vdash C},\quad (\mathrm{R}{\leftarrow})\ \frac{\Gamma, A \vdash B}{\Gamma \vdash B \leftarrow A},$$

$$(\mathrm{L}{\otimes})\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, A \otimes B, \Gamma' \vdash C},\quad (\mathrm{R}{\otimes})\ \frac{\Gamma \vdash A;\ \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B},$$

$$(\mathrm{Cut})\ \frac{\Gamma, A, \Gamma' \vdash B;\ \Delta \vdash A}{\Gamma, \Delta, \Gamma' \vdash B}.$$

In this paper A, B, C (also with extras) range over propositional formulas, built from propositional variables ($p, q, r, \ldots$) by means of logical connectives (here $\rightarrow, \leftarrow, \otimes$), and Γ, Δ range over finite sequences of formulas. So, the Lambek calculus operates on sequents $\Gamma \vdash A$. One assumes $\Gamma \neq \Lambda$, where Λ stands for the empty sequence, in rules (R$\rightarrow$) and (R$\leftarrow$). Accordingly, all sequents derivable in this calculus have nonempty antecedents (there are no theorems!). Constants $\rightarrow, \leftarrow, \otimes$ will be called right implication, left implication and product (multiplicative conjunction), respectively. The product-free fragment of the Lambek calculus will be denoted by **L** and the full Lambek calculus by $\mathbf{L}[\otimes]$. A similar notation will be used for other enrichments of **L** (and other systems). For instance, $\mathbf{L}[\wedge]$ is **L** enriched with (additive) conjunction $\wedge$, and the rules for $\wedge$ are:

$$(\mathrm{L}\wedge 1)\ \frac{\Gamma, A, \Gamma' \vdash C}{\Gamma, A \wedge B, \Gamma' \vdash C},\ (\mathrm{L}\wedge 2)\ \frac{\Gamma, B, \Gamma' \vdash C}{\Gamma, A \wedge B, \Gamma' \vdash C},$$

$$(\mathrm{R}\wedge)\ \frac{\Gamma \vdash A;\ \Gamma \vdash B}{\Gamma \vdash A \wedge B}.$$

Of course, $\mathbf{L}[\otimes, \wedge]$ is **L** enriched with both $\otimes$ and $\wedge$, and so on.

Most general algebraic frames for $\mathbf{L}[\otimes]$ are *residuated semigroups*, i.e. structures $(M, \circ, \Rightarrow, \Leftarrow, \leq)$ such that $(M, \circ)$ is a semigroup, $(M, \leq)$ is a poset, and $\Rightarrow, \Leftarrow$ are binary operations on M, satisfying the equivalences:

$$(\mathrm{RES})\ y \leq x \Rightarrow z \text{ iff } x \circ y \leq z \text{ iff } x \leq z \Leftarrow y,$$

for all $x, y, z \in M$. A model is a pair (M, μ) such that M is a residuated semigroup, and μ is an assignment of elements of M to propositional variables, which is extended to all formulas by setting:

$$\mu(A \rightarrow B) = \mu(A) \Rightarrow \mu(B),\ \mu(A \leftarrow B) = \mu(A) \Leftarrow \mu(B),$$

$$\mu(A \otimes B) = \mu(A) \circ \mu(B).$$

Sequent $A_1 \ldots A_n \vdash B$ is said to be *true* in this model, if:

$$\mu(A_1) \circ \cdots \circ \mu(A_n) \leq \mu(B).$$

By the standard method of Lindenbaum algebras, one proves that $\mathbf{L}[\otimes]$ is complete with respect to these models, that means, the sequents derivable in

$\mathbf{L}[\otimes]$ are precisely the sequents true in all models of that kind. Strong completeness also holds: for any set Φ, of sequents with nonempty antecedents, the sequents derivable in $\mathbf{L}[\otimes]$ enriched with all sequents from Φ as new axioms are precisely the sequents true in all models in which all sequents from Φ are true.

Let $\mathcal{M} = (M, \cdot)$ be a semigroup. On the powerset $P(M)$ one defines operations $\circ, \Rightarrow, \Leftarrow$ by setting:

$$X \circ Y = \{xy : x \in X,\ y \in Y\},$$

$$X \Rightarrow Y = \{z \in M : (\forall x \in X) xz \in Y\},$$

$$X \Leftarrow Y = \{z \in M : (\forall y \in Y) zy \in X\},$$

for $X, Y \subseteq M$ (we write xy for $x \cdot y$). The structure

$$P(\mathcal{M}) = (P(M), \circ, \Rightarrow, \Leftarrow, \subseteq)$$

is a residuated semigroup; it will be called *the powerset frame over* $\mathcal{M}$. Models $(P(\mathcal{M}), \mu)$ will be called *powerset models*. In powerset models, one interprets formulas with $\wedge$ as follows:

$$\mu(A \wedge B) = \mu(A) \cap \mu(B).$$

It is obvious that $\mathbf{L}[\otimes, \wedge]$ is sound with respect to powerset models, and it has been shown in [5] that this calculus is also strongly complete with respect to powerset models.

Standard linguistic frames for the Lambek calculus are powerset frames over (finitely generated) free semigroups. Thus, $\mathcal{M} = (V^+, \cdot)$, where V is a nonempty finite set (the alphabet or lexicon), V^+ denotes the set of all nonempty finite sequences on V, and $\cdot$ is concatenation. Elements of $P(V^+)$ are $\Lambda-$free languages on V. In Categorial Grammar, a model $(P(\mathcal{M}), \mu)$, with $\mathcal{M}$ as above, is interpreted as the hierarchy of syntactic categories of a language. In this context, only finitely many variables are employed, and the assignment μ sends them to basic grammatical categories, e.g. $\mu(s)$ is the set of sentences, $\mu(n)$ is the set of noun phrases, and so on. $\mu(A \rightarrow B)$ (resp. $\mu(B \leftarrow A)$) consists of so-called left-looking (resp. right-looking) functors from category $\mu(A)$ to category $\mu(B)$, and $A \rightarrow B$ (resp. $B \leftarrow A$) is a type of these functors. For instance, a verb phrase is a functor of type $n \rightarrow s$, and a transitive verb phrase is a functor of type $(n \rightarrow s) \leftarrow n$. $\mathbf{L}[\wedge]$ is strongly complete with respect to powerset models over free semigroups [5], but $\mathbf{L}[\otimes]$ is merely weakly complete with respect to this semantics (see Pentus [15]).

Let $\mathcal{M} = (M, \cdot)$ be a semigroup, and let K be a nonempty subset of M, satisfying the condition:

(K) for all $x, y, z \in K$, if $xyz \in K$, then $xy \in K$ iff $yz \in K$.

On the powerset $P(K)$ we define operations $\circ, \Rightarrow, \Leftarrow$ as follows:

$$X \circ Y = \{xy \in K : x \in X,\ y \in Y\},$$

$$X \Rightarrow Y = \{z \in K : (\forall x \in X)(\text{if } xz \in K \text{ then } xz \in Y)\},$$

$$X \Leftarrow Y = \{z \in K : (\forall y \in Y)(\text{if } zy \in K \text{ then } zy \in X)\},$$

for $X, Y \subseteq K$. The structure

$$P_K(\mathcal{M}) = (P(K), \circ, \Rightarrow, \Leftarrow, \subseteq)$$

is called *the restricted powerset frame over* $\mathcal{M}$ with *universe* K. $P_K(\mathcal{M})$ is a residuated semigroup: the associativity of $\circ$ follows from (K), and the proof of (RES) is easy. Since powerset frames are particular restricted powerset frames ($K = M$), then $\mathbf{L}[\otimes, \wedge]$ is strongly complete with respect to restricted powerset models.

We are especially interested in restricted powerset frames over free semigroups $\mathcal{M} = (V^+, \cdot)$ such that K is closed under nonempty intervals, which is expressed by the condition:

(K') for all $x, y \in V^+$, if $xy \in K$ then $x \in K$ and $y \in K$.

Clearly, (K') entails (K) but not conversely. In [7] I argue for the usefullness of these frames in natural language modelling. The key idea is that K consists of those expressions on V which do not contain evident grammatical nonsenses.

By $\mathcal{F}$ we denote the class of all restricted powerset frames over finitely generated free semigroups whose universe K is finite and satisfies (K'). Models $(P_K(\mathcal{M}), \mu)$ such that $P_K(\mathcal{M}) \in \mathcal{F}$ will be called $\mathcal{F}$-models. Each $\mathcal{F}$-model is essentially a finite model and can uniquely be represented by the pair (K, μ). In section 3 we prove that $\mathbf{L}[\wedge]$ is complete with respect to $\mathcal{F}$-models. In section 4 we effectively define a class of finite sets Φ, of sequents with nonempty antecedents, such that the axiomatic extension of $\mathbf{L}[\wedge]$ by Φ is also complete with respect to $\mathcal{F}$-models. That yields a proof of decidability of these finite axiomatic extensions of $\mathbf{L}[\wedge]$; the decidability of pure logics, considered above, follows from the cut-elimination theorem, but this theorem does not hold for axiomatic extensions of these logics. Caution: not all finite axiomatic extensions of **L** are decidable [4], hence one cannot expect they all are complete with respect to $\mathcal{F}$-models.

Section 5 provides analogous results for other systems: system **L1** which arises from **L**, after one has dropped the constraint $\Gamma \neq \Lambda$ in rules (R$\rightarrow$), (R$\leftarrow$) (enrichments of **L1** are fragments of noncommutative linear logics of Abrusci [1]), system **L1**$[\wedge]$ as well as commutative versions of these systems (the commutative **L1** equals the BCI-logic and is a fragment of MALL). The proof for **L**$[\wedge]$ has been arranged in such a form that all further modifications are straightforward.

In section 6 we show that our methods also yield the result of Lafont [11]; we use a syntactic interpretation of MALL in system **L1E**$[\wedge, \top]$ and the completeness of the latter with respect to finite restricted powerset models over commutative monoids. Quite likely, the completeness of **L1**$[\otimes, \wedge]$ and richer noncommutative linear logics with respect to finite noncommutative phase space models also follows from our completeness theorems, by an analogous syntactic interpretation, but we have not checked the details. Although the proof of Lafont [11] seems to be more natural than my proofs, it seems to be applicable to neither systems like **L** in which all derivable sequents have nonempty antecedents, nor axiomatic extensions of pure logics.

2 Barriers

First, we recall the construction of a canonical model for **L**, as given in [4]. Let T be a nonempty finite set of formulas, closed under subformulas. We consider the free semigroup $\mathcal{M} = (T^+, \cdot)$. An assignment μ of variables in $P(M)$ is defined as follows:

$$\text{(CM1) } \mu(p) = \{\Gamma \in T^+ : \Gamma \vdash_{\mathbf{L}} p\}.$$

By induction on the complexity of A, one easily proves:

$$\text{(CM2) } \mu(A) = \{\Gamma \in T^+ : \Gamma \vdash_{\mathbf{L}} A\},$$

for all $A \in T$. The canonical equivalence

$$\text{(CE) } \Gamma \vdash_{\mathbf{L}} A \text{ iff } \Gamma \vdash A \text{ is true in } (P(\mathcal{M}), \mu),$$

for all $\Gamma \in T^+$, $A \in T$, is shown in the following way. The implication ($\Rightarrow$) holds by the soundness of **L** with respect to powerset models. For ($\Leftarrow$), assume $\Gamma \not\vdash_{\mathbf{L}} A$. Let $\Gamma = A_1 \ldots A_n$. We have $A_i \in \mu(A_i)$, by (Id) and (CM2), but $A_1 \ldots A_n \notin \mu(A)$, again by (CM2). Consequently, $\Gamma \vdash A$ is not true in $(P(\mathcal{M}), \mu)$. Clearly, this construction yields the completeness of $\mathbf{L}$ with respect to powerset models over finitely generated free semigroups.

Let us modify the above construction to obtain an $\mathcal{F}$-model. By $\ell(\Gamma)$ we denote the length of Γ. We fix an integer $n > 0$ and define $K_n = \{\Gamma \in T^+ : \ell(\Gamma) \leq n\}$. We consider the $\mathcal{F}$-model (K_n, μ) such that:

$$\text{(CM1')}\ \mu(p) = \{\Gamma \in K_n : \Gamma \vdash_{\mathbf{L}} p\},$$

for every propositional variable p. Unfortunately, the analogue of (CM2):

$$\text{(CM2')}\ \mu(A) = \{\Gamma \in K_n : \Gamma \vdash_{\mathbf{L}} A\},$$

does not hold, in general. Take $n = 2$. We assume T contain all the formulas mentioned below. Denote $A = p \rightarrow (p \rightarrow p)$. All strings $\Gamma \in T^+$ such that $\ell(\Gamma) = 1$ belong to $\mu(A)$. Thus, $p \in \mu(A)$, although $p \not\vdash_{\mathbf{L}} A$. Denote $B = A \rightarrow p$. Since $p, B \not\vdash_{\mathbf{L}} p$, then the antecedent of the latter sequent is not in $\mu(p)$, by (CM1'), and consequently, $B \notin \mu(B)$, although $B \vdash_{\mathbf{L}} B$. Moral: in models defined by (CM1'), 'long' sequences Γ can belong to $\mu(A)$, if even $\Gamma \vdash A$ is not derivable in $\mathbf{L}$, and 'short' sequences Γ need not belong to $\mu(A)$, if even $\Gamma \vdash A$ is derivable in $\mathbf{L}$. Similar patologies would appear, if instead of $\ell(\Gamma)$ one used another standard complexity measure, e.g. $\rho(\Gamma)$ equal to the number of occurrences of variables in Γ.

Accordingly, to construct an appropriate canonical $\mathcal{F}$-model we must go in a more subtle way. The idea is to introduce some fine complexity measures, called barriers. We define them in a form which is appropriate for all systems under consideration and finite axiomatic extensions of these systems.

Sequents of the form $A \vdash B$ are said to be *simple*. In $\mathbf{L}$, $\mathbf{L}[\wedge]$ and $\mathbf{L}[\otimes, \wedge]$, sequents:

(S1) $A_1 \ldots A_n \vdash B$,

(S2) $A_n \vdash A_{n-1} \rightarrow (A_{n-2} \rightarrow \cdots \rightarrow (A_1 \rightarrow B) \ldots)$,

are deductively equivalent. Also, for every model $(\mathcal{M}, \mu)$, (S1) is true in $(\mathcal{M}, \mu)$ iff (S2) is so. Accordingly, we may confine ourselves to simple sequents in some definitions and arguments concerning completeness.

Let Φ, Ψ be two finite sets of simple sequents such that $\Phi \cap \Psi = \emptyset$. By a *concrete formula* in (Φ, Ψ) we mean a single occurrence of a subformula in a sequent from $\Phi \cup \Psi$. By $F(\Phi, \Psi)$ we denote the set of all concrete formulas in (Φ, Ψ). Each element of $F(\Phi, \Psi)$ is qualified as *positive* or *negative* according to the following recursive conditions:

(PN1) if $A \vdash B$ is in Φ, then A is positive, and B is negative; if $A \vdash B$ is in Ψ, then A is negative, and B is positive.

(PN2) if $A \rightarrow B$ or $B \leftarrow A$ is positive (resp. negative), then A is negative (resp. positive), and B is positive (resp. negative),

(PN3) if $A \wedge B$ is positive (resp. negative), then A and B are positive (resp. negative).

In this definition and often in the sequel, A, B, C stand for concrete formulas; if $A \rightarrow B$ is a concrete formula, then A, B denote the particular concrete subformulas of $A \rightarrow B$, and similarly for other cases.

By *barriers* for (Φ, Ψ) we mean a pair L, U, of functions from $F(\Phi, \Psi)$ to the set of natural numbers, satisfying the following conditions:

(B0.1) $L(A) \leq U(B)$, for all $A, B \in F(\Phi, \Psi)$,

(B0.2) if $A \vdash B$ is in Φ, then $L(A) \geq L(B)$ and $U(A) \geq U(B)$,

(B1) $0 < L(p') = L(p'')$ and $U(p') = U(p'')$, for every variable p which occurs in $\Phi \cup \Psi$ and all particular occurrences p', p'' of this variable,

(B2) $L(A \rightarrow B) = \max(0, L(B) - L(A))$, if $A \rightarrow B$ is positive, and similarly for $B \leftarrow A$,

(B3) $L(A \rightarrow B) = \max(1, L(B) - L(A), U(B) - U(A))$, if $A \rightarrow B$ is negative, and similarly for $B \leftarrow A$,

(B4) $U(A \rightarrow B) = U(B) - L(A)$, if $A \rightarrow B$ is in $F(\Phi, \Psi)$, and similarly for $B \leftarrow A$,

(B5) $L(A \wedge B) = \max(L(A), L(B))$, if $A \wedge B$ is in $F(\Phi, \Psi)$,

(B6) $U(A \wedge B) = \min(U(A), U(B))$, if $A \wedge B$ is positive,

(B7) $U(A \wedge B) = \max(U(A), U(B))$, if $A \wedge B$ is negative.

Observe that $L(A) > 0$, for all negative $A \in F(\Phi, \Psi)$. We need the following lemma.

Lemma 1 (Extension lemma)
Assume that L, U are barriers for (Φ, Ψ), $\Psi \subseteq \Psi'$, Ψ' is finite and disjoint with Φ. Then, one effectively finds a pair L', U', of barriers for (Φ, Ψ').

PROOF. We have $F(\Phi, \Psi) \subseteq F(\Phi, \Psi')$. For any variable p which occurs in Ψ' but not in $\Phi \cup \Psi$, we set $L^{\sim}(p') = U^{\sim}(p') = 1$, for all occurrences p' of p in $\Phi \cup \Psi'$. We also set $L^{\sim}(q') = L(q')$ and $U^{\sim}(q') = U(q')$, for all concrete variables $q' \in F(\Phi, \Psi)$. Thus, functions $L^{\sim}, U^{\sim}$ are defined for all concrete variables in $F(\Phi, \Psi')$ and satisfy (B1). By (B2)-(B7), we uniquely extend them to be defined on all concrete formulas in $F(\Phi, \Psi')$. The extended functions $L^{\sim}, U^{\sim}$ fulfil conditions (B0.2) and (B1)-(B7), but they need not fulfil (B0.1); further, $U^{\sim}$ can admit negative values. We define:

$$k = \max\{L^{\sim}(A) - U^{\sim}(B) : A, B \in F(\Phi, \Psi')\},$$

$$L'(A) = L^{\sim}(A),\ U'(A) = U^{\sim}(A) + k,\ \text{for } A \in F(\Phi, \Psi').$$

By indution of the complexity of (concrete) formulas, one easily shows that functions L', U' also fulfil (B0.2) and (B1)-(B7). They also fulfil (B0.1), since:

$$L'(A) = L^{\sim}(A) = U^{\sim}(B) + L^{\sim}(A) - U^{\sim}(B) \leq U^{\sim}(B) + k = U'(B),$$

for all $A, B \in F(\Phi, \Psi')$. Consequently, $U'(B) \geq 0$, for all $B \in F(\Phi, \Psi')$, and L', U' are barriers for (Φ, Ψ'). □

3 Fmp for $\mathbf{L}[\wedge]$

We fix a sequent $\overline{A} \vdash \overline{B}$, not derivable in $\mathbf{L}[\wedge]$. We shall construct an $\mathcal{F}$-model (K, μ) such that $\overline{A} \vdash \overline{B}$ is not true in (K, μ). We shortly write $\mathbf{S}$ for $\mathbf{L}[\wedge]$.

Put $\Phi = \emptyset$, $\Psi = \{\overline{A} \vdash \overline{B}\}$. There is a pair L, U, of barriers for (Φ, Ψ). For the pair $\emptyset, \emptyset$, of empty functions, are barriers for $(\Phi, \emptyset)$, and they can be extended, by lemma 1.

We define T as the set of all $A \in F(\Phi, \Psi)$ such that $L(A) > 0$. Notice that all negative $A \in F(\Phi, \Psi)$ belong to T. So, T is nonempty and finite. For $A \in F(\Phi, \Psi)$, by A^o we denote the (nonconcrete) formula represented by the particular occurrence A. Also, for $\Gamma = A_1 \ldots A_k$, we set $\Gamma^o = A_1^o \ldots A_k^o$. Define:

$$n = \max\{U(A) : A \in F(\Phi, \Psi)\},$$

$$L(A_1 \ldots A_k) = L(A_1) + \cdots + L(A_k),\ \text{for } A_1 \ldots A_k \in T^+,$$

$$K = \{\Gamma \in T^+ : L(\Gamma) \leq n\},$$

$$\mu(p^o) = \{\Gamma \in K : (L(p) \leq L(\Gamma) \text{ and } \Gamma^o \vdash_{\mathbf{S}} p^o) \text{ or } L(\Gamma) > U(p)\},$$

for all variables p^o occurring in $\Phi \cup \Psi$; for the remaining variables, the value of μ is not essential. Caution: the last definition is correct, by (B1).

Lemma 2 (Canonical model lemma)
For all $C \in F(\Phi, \Psi)$, $\Gamma \in K$, there hold the following conditions:
(C1) if C is positive, $L(\Gamma) \leq U(C)$, and $\Gamma \in \mu(C^o)$, then $\Gamma^o \vdash_{\mathbf{S}} C^o$,
(C2) if C is positive, and $L(\Gamma) < L(C)$, then $\Gamma \notin \mu(C^o)$,
(C3) if C is negative, $L(C) \leq L(\Gamma)$, and $\Gamma^o \vdash_{\mathbf{S}} C^o$, then $\Gamma \in \mu(C^o)$,
(C4) if C is negative, and $L(\Gamma) > U(C)$, then $\Gamma \in \mu(C^o)$.

PROOF. Simultaneous induction on the complexity of C. If C is a concrete variable, then (C1)-(C4) hold, by the definition of μ.

We consider the case $C = A \rightarrow B$ (for $C = B \leftarrow A$, the reasoning is dual).

We prove (C1). Assume C be positive, $L(\Gamma) \leq U(C)$, and $\Gamma \in \mu(C^o)$. Clearly, A is negative, B is positive, and $C^o = A^o \rightarrow B^o$. Since $A \in T$ and $L(A) \leq U(A) \leq n$, then $A \in K$. Since $A^o \vdash_{\mathbf{S}} A^o$ and $L(A) \leq L(A)$, then $A \in \mu(A^o)$, by (C3). We have:

$$L(A \cdot \Gamma) = L(A) + L(\Gamma) \leq L(A) + U(B) - L(A) = U(B),$$

by (B4). Consequently, $A \cdot \Gamma \in K$, hence $A \cdot \Gamma \in \mu(B^o)$, and $A^o, \Gamma^o \vdash_{\mathbf{S}} B^o$, by (C1). Using (R$\rightarrow$), we obtain $\Gamma^o \vdash_{\mathbf{S}} C^o$.

We prove (C2). Assume C be positive, and $L(\Gamma) < L(C)$. Then, $L(C) > 0$, and consequently, $L(C) = L(B) - L(A)$, by (B2). As above, $A \in \mu(A^o)$. We have:

$$L(A \cdot \Gamma) = L(A) + L(\Gamma) < L(A) + L(B) - L(A) = L(B).$$

Consequently, $A \cdot \Gamma \in K$, but $A \cdot \Gamma \notin \mu(B^o)$, by (C2) ($B$ is positive). Thus, $\Gamma \notin C^o$.

We prove (C3). Assume C be negative, $L(C) \leq L(\Gamma)$, and $\Gamma^o \vdash_{\mathbf{S}} C^o$. Then, A is positive, B is negative, and again $C^o = A^o \rightarrow B^o$. Take $\Delta \in \mu(A^o)$ such that $\Gamma \cdot \Delta \in K$. We consider two cases.

(I) $L(\Delta) > U(A)$. Then:

$$L(\Gamma \cdot \Delta) = L(\Gamma) + L(\Delta) > U(B) - U(A) + U(A) = U(B),$$

since $L(C) \geq U(B) - U(A)$, by (B3). Consequently, $\Gamma \cdot \Delta \in \mu(B^o)$, by (C4).

(II) $L(\Delta) \leq U(A)$. By (C1), $\Delta^o \vdash_{\mathbf{S}} A^o$, and consequently, $\Gamma^o, \Delta^o \vdash_{\mathbf{S}} B^o$, by (Cut) and the **S**-derivable sequent $A^o, A^o \to B^o \vdash B^o$. Now:

$$L(\Gamma \cdot \Delta) = L(\Gamma) + L(\Delta) \geq L(B) - L(A) + L(A) = L(B),$$

since $L(C) \geq L(B) - L(A)$, by (B3), and $L(\Delta) \geq L(A)$, by (C2). Consequently, $\Gamma \cdot \Delta \in \mu(B^o)$, by (C3).

We have shown $\Gamma \in \mu(C^o)$, as required.

We prove (C4). Assume C be negative, and $L(\Gamma) > U(C)$. Take $\Delta \in \mu(A^o)$ such that $\Gamma \cdot \Delta \in K$. We have $L(A) \leq L(\Delta)$, by (C2), hence:

$$L(\Gamma \cdot \Delta) = L(\Gamma) + L(\Delta) > U(B) - L(A) + L(A) = U(B),$$

by (B4), and consequently, $\Gamma \cdot \Delta \in \mu(B^o)$, by (C4). We have shown $\Gamma \in \mu(C^o)$.

We consider the case $C = A \wedge B$.

We prove (C1). Assume C be positive, $L(\Gamma) \leq U(C)$, and $\Gamma \in \mu(C^o)$. By (B6), $L(\Gamma) \leq U(A)$ and $L(\Gamma) \leq U(B)$. Clearly, A and B are positive, and $C^o = A^o \wedge B^o$. Using (C1), we obtain $\Gamma^o \vdash A^o$ and $\Gamma^o \vdash B^o$, hence $\Gamma^o \vdash C^o$, by (R$\wedge$).

We prove (C2). Assume C be positive, and $L(\Gamma) < L(C)$. By (B5), either $L(\Gamma) < L(A)$, or $L(\Gamma) < L(B)$. In the former case, $\Gamma \not\in \mu(A^o)$, by (C2), and in the latter case, $\Gamma \not\in \mu(B^o)$, again by (C2). Consequently, $\Gamma \not\in \mu(C^o)$.

We prove (C3). Assume C be negative, $L(C) \leq L(\Gamma)$, and $\Gamma^o \vdash_{\mathbf{S}} C^o$. Then, A and B are negative, $L(A) \leq L(\Gamma)$ and $L(B) \leq L(\Gamma)$, by (B5). We also have $\Gamma^o \vdash_{\mathbf{S}} A^o$ and $\Gamma^o \vdash_{\mathbf{S}} B^o$, by (Cut) and **S**-derivable sequents $A^o \wedge B^o \vdash A^o$, $A^o \wedge B^o \vdash B^o$. Using (C3), we obtain $\Gamma \in \mu(A^o)$, $\Gamma \in \mu(B^o)$, which yields $\Gamma \in \mu(C^o)$.

We prove (C4). Assume C be negative, and $L(\Gamma) > U(C)$. By (B7), $L(\Gamma) > U(A)$ and $L(\Gamma) > U(B)$. Using (C4), we obtain $\Gamma \in \mu(A^o)$ and $\Gamma \in \mu(B^o)$, and consequently, $\Gamma \in \mu(C^o)$. $\Box$

Theorem 1 *For all formulas $\overline{A}, \overline{B}$ of $\mathbf{S} = \mathbf{L}[\wedge]$, $\overline{A} \vdash_{\mathbf{S}} \overline{B}$ if, and only if, $\overline{A} \vdash \overline{B}$ is true in all $\mathcal{F}$-models.*

Proof. Implication ($\Rightarrow$) holds, by soundness. To prove ($\Leftarrow$), assume $\overline{A} \vdash \overline{B}$ be not derivable in **S**. Construct an $\mathcal{F}$-model (K, μ), as in lemma 2. Since $\overline{A} \in F(\Phi, \Psi)$ is negative, $L(\overline{A}) \leq L(\overline{A})$ and $\overline{A} \vdash_{\mathbf{S}} \overline{A}$, then $\overline{A} \in \mu(\overline{A})$, by (C3). Also, $\overline{B} \in F(\Phi, \Psi)$ is positive, $L(\overline{A}) \leq U(\overline{B})$, by (B0.1), and $\overline{A} \not\vdash_{\mathbf{S}} \overline{B}$, and consequently, $\overline{A} \notin \mu(\overline{B})$, by (C1). We have shown $\mu(\overline{A}) \not\subseteq \mu(\overline{B})$, as desired. Caution: in this proof, we do not discriminate between formulas $\overline{A}, \overline{B}$ and their occurrences in sequent $\overline{A} \vdash \overline{B}$. $\Box$

4 Fmp for finite axiomatic extensions of $\mathbf{L}[\wedge]$

Let Φ be a finite set of simple sequents. The set Φ is said to be *bounded*, if there exists a pair L, U, of barriers for $(\Phi, \emptyset)$. One easily checks that, for any finite set Φ, the condition 'Φ is bounded' can effectively be written as a sentence of Presburger arithmetic (arithmetic without multiplication). Consequently, one can effectively verify whether any finite set Φ is bounded and, if so, effectively find a pair L, U, of barriers for $(\Phi, \emptyset)$.

We fix Φ and write $\mathbf{S}$ for the system $\mathbf{L}[\wedge]$ enriched with all sequents from Φ as new axioms. We prove:

Theorem 2 *If Φ is bounded, then, for all formulas $\overline{A}, \overline{B}$, of $\mathbf{L}[\wedge]$, $\overline{A} \vdash_{\mathbf{S}} \overline{B}$ if, and only if, $\overline{A} \vdash \overline{B}$ is true in all $\mathcal{F}$-models (K, μ) such that each sequent from Φ is true in (K, μ).*

PROOF. Assume Φ be bounded. We fix a pair L_0, U_0, of barriers for $(\Phi, \emptyset)$. As above, it suffices to prove the implication ($\Leftarrow$) of the theorem. Let $\overline{A} \vdash \overline{B}$ be not derivable in $\mathbf{S}$. We denote $\Psi = \{\overline{A} \vdash \overline{B}\}$. By lemma 1, there exists a pair L, U, of barriers for (Φ, Ψ). We construct an $\mathcal{F}$-model (K, μ), as in lemma 2, and prove (C1)-(C4) in exactly the same way. Also, we prove that $\overline{A} \vdash \overline{B}$ is not true in (K, μ), by the same argument as in the proof of theorem 1. It remains to show that all sequents from Φ are true in (K, μ). Let $A \vdash B$ belong to Φ. Then, $A, B \in F(\Phi, \Psi)$, A is positive, and B is negative. Obviously $A^o \vdash_{\mathbf{S}} B^o$. Let $\Gamma \in \mu(A^o)$. We consider two cases.

CASE 1. $L(\Gamma) > U(A)$. By (B0.2), $L(\Gamma) > U(B)$, hence $\Gamma \in \mu(B^o)$, by (C4).

CASE 2. $L(\Gamma) \leq U(A)$. Then, $\Gamma^o \vdash_{\mathbf{S}} A^o$, by (C1). Using (Cut), we obtain $\Gamma^o \vdash_{\mathbf{S}} B^o$. Also, $L(A) \leq L(\Gamma)$, by (C2), and $L(B) \leq L(A)$, by (B0.2), which yields $L(B) \leq L(\Gamma)$. Using (C3), we obtain $\Gamma \in \mu(B^o)$.

We have shown $\mu(A^o) \subseteq \mu(B^o)$, which means that $A^o \vdash B^o$ is true in (K, μ). $\Box$

EXAMPLES.

Each finite set of 'context-free' sequents $p_1 \ldots p_n \vdash q$ is bounded. We call these sequents 'context-free', since they correspond to production rules $q \mapsto p_1 \ldots p_n$ of Context-Free Grammars (variables represent nonterminal symbols). There are interesting applications of 'context-free' extensions of the Lambek calculus; for instance, the Lambek calculus can transform the given Context-Free Grammar into a Context-Free Grammar in the Greibach Normal Form [8]. We represent each sequent $p_1 \ldots p_n \vdash q$ as the equivalent simple sequent:

$$p_n \vdash p_{n-1} \to (p_{n-2} \to \cdots \to (p_1 \to q) \ldots).$$

Let Φ consist of all simple sequents obtained in this way from the given finite set of 'context-free' sequents. Define $L(p) = 1$, and $U(p) = m + 1$, where m is the maximal length of antecedents of 'context-free' sequents, for all occurrences p of variables in Φ, and extend L, U to all concrete formulas in $F(\Phi, \emptyset)$, using (B2)-(B7). One easily shows that L, U are a pair of barriers for $(\Phi, \emptyset)$. Consequently, each finite 'context-free' extension of $\mathbf{L}[\wedge]$ is complete with respect to $\mathcal{F}$-models, hence it is decidable. (The decidability of such systems can also be shown by an appropriate cut-elimination theorem; see [4].)

Take $\Phi = \{p \to q \vdash r\}$, where p, q, r are distinct variables. Define $L(p) = L(r) = 1$, $L(q) = 2$, $U(p) = U(r) = 2$, $U(q) = 3$. Since $p \to q$ is positive, we obtain $L(p \to q) = 1$, by (B2), and $U(p \to q) = 2$, by (B4). Thus, L, U are barriers for $(\Phi, \emptyset)$, and Φ is bounded.

Take $\Phi = \{p \to p \vdash p\}$. There are no barriers for $(\Phi, \emptyset)$. For $U(p \to p) = U(p) - L(p) < U(p)$, hence (B0.2) cannot be satisfied. Actually, $\mathbf{L}[\wedge]$ enriched by Φ is *not* complete with respect to $\mathcal{F}$-models. We must have $\mu(p) = K$, for every $\mathcal{F}$-model (K, μ) such that $\mu(p \to p) \subseteq \mu(p)$. For all maximal sequences in K belong to $\mu(p \to p)$, hence they belong to $\mu(p)$. Therefore, all 'almost maximal' sequences in K also belong to $\mu(p \to p)$, hence to $\mu(p)$, and so on. Accordingly, if $\mu(p \to p) \subseteq \mu(p)$, then also $\mu(q) \subseteq \mu(p)$, for any variable q, but $q \vdash p$ does not follow from $p \to p \vdash p$ within $\mathbf{L}[\wedge]$ (use the strong completeness of $\mathbf{L}[\wedge]$ with respect to powerset models over free semigroups and construct an appropriate model of that kind).

On the other hand, the system $\mathbf{L}[\wedge]$ enriched with $p \to p \vdash p$ is decidable [4]. It is still possible that this system is complete with respect to a more general class of finite models, but we know no proof of this hypothesis. Further, the boundedness of Φ is merely a sufficient condition for the completeness of $\mathbf{L}[\wedge]$ enriched with Φ with respect to $\mathcal{F}$-models; it would be nice to find out a necessary and sufficient condition.

Of course, all the results obtained above also hold for $\mathbf{L}$ and its finite axiomatic extensions, and they have been proved for $\mathbf{L}$ in [4] by applying another notion of barriers (defined on formulas instead of concrete formulas) which cannot be used for the case of $\mathbf{L}[\wedge]$. The method presented above (with a slightly more complicated condition substituting (B2) and more complication in the proof of lemma 2) has been elaborated in [9], but axiomatic extensions have not been handled there (hence, condition (B0.2) has been abandoned). In [4, 9], cofinite powerset models have been applied instead of $\mathcal{F}$-models.

5 Fmp for other systems

Recall that $\mathbf{L1}$ arises from $\mathbf{L}$ by admitting $\Gamma = \Lambda$ in rules (R$\rightarrow$), (R$\leftarrow$), and similarly for $\mathbf{L1}[\wedge]$. Most general algebraic frames for $\mathbf{L1}$ are *residuated monoids*, i.e. structures $(M, \circ, \Rightarrow, \Leftarrow, 1, \leq)$ such that $(M, \circ, \Rightarrow, \Leftarrow, \leq)$ is a residuated semigroup, and $(M, \circ, 1)$ is a monoid (that means, $1 \in M$ satisfies $1 \circ x = x \circ 1 = x$, for all $x \in M$). The truth of sequent $\Gamma \vdash A$ in model $(\mathcal{M}, \mu)$ is defined as above with the following addition: $\vdash A$ is true in (M, μ), if $1 \leq \mu(A)$. Each sequent is deductively (in $\mathbf{L1}[\wedge]$) and semantically (in every model) equivalent to a sequent with empty antecedent.

If $\mathcal{M} = (M, \cdot, 1)$ is a monoid, then the structure:

$$P(\mathcal{M}) = (P(M), \circ, \Rightarrow, \Leftarrow, \{1\}, \subseteq),$$

where $\circ, \Rightarrow, \Leftarrow$ are defined as for powerset frames over semigroups, is a residuated monoid; it is called *the powerset frame over* $\mathcal{M}$.

Restricted powerset frames over a monoid $\mathcal{M} = (M, \cdot, 1)$ are defined as structures:

$$P_K(\mathcal{M}) = (P(K), \circ, \Rightarrow, \Leftarrow, \{1\}, \subseteq),$$

such that $\emptyset \neq K \subseteq M$ satisfies (K) and $1 \in K$ (operations are defined as above). They are residuated monoids. The class $\mathcal{F}_1$ consists of restricted powerset frames over free monoids $(V^\star, \cdot, \Lambda)$ (here, $V^\star$ is the set of all finite sequences on V) with the universe K being finite and satisfying (K') and $\Lambda \in K$.

$\mathbf{L1}[\wedge, \otimes]$ is strongly complete with respect to powerset models over arbitrary monoids [5], and evidently, with respect to arbitrary residuated monoids. We prove that $\mathbf{L1}[\wedge]$ is complete with respect $\mathcal{F}_1$-models. It suffices to consider sequents $\vdash B$. We admit sequents of this form can occur in Ψ (not in Φ). We also set $L(\Lambda) = 0$.

Theorem 3 *For every formula $\overline{B}$ of $\mathbf{L1}[\wedge]$, $\vdash \overline{B}$ is derivable in this system if, and only if, $\vdash \overline{B}$ is true in all $\mathcal{F}_1$-models.*

PROOF. Take $\Psi = \{\vdash B\}$. As above, we find a pair L, U, of barriers for (Φ, Ψ), where $\Phi = \emptyset$. We define T as above and consider the $\mathcal{F}_1$-model (K, μ) such that:

$$K = \{\Gamma \in T^\star : L(\Gamma) \leq n\},$$

and the remaining notions are defined as above. The proof of lemma 2 can be repeated without changes. We prove the implication $(\Leftarrow)$ of the theorem. Assume $\vdash \overline{B}$ be not derivable in $\mathbf{L1}[\wedge]$. Since $L(\Lambda) = 0 \leq U(\overline{B})$ and $\overline{B}$ is positive, we infer $\Lambda \notin \mu(\overline{B})$, by (C1). Consequently, $\vdash \overline{B}$ is not true in (K, μ). $\Box$

An analogue of theorem 2 for $\mathbf{L1}[\wedge]$ can be proved in the same way. Notice that it is somewhat artificially restricted to sets Φ consisting of simple sequents; that excludes sequents $\vdash p$ (p is a variable) which are not deductively equivalent to any simple sequent. To regard these sequents our method must be modified. The condition (B1) is to be replaced by:

(B1') $L(p) = 0$, for all positive $p \in F(\Phi, \Psi)$, $L(p) > 0$, for all negative $p \in F(\Phi, \Psi)$, and $U(p') = U(p'')$, for all occurrences p', p'' of the same variable p occurring in $\Phi \cup \Psi$,

and the definition of $\mu(p^o)$ is simplified:

$$\mu(p^o) = \{\Gamma \in K : \Gamma^o \vdash_{\mathbf{S}} p^o \text{ or } L(\Gamma) > U(p)\}.$$

Then, the analogue of lemma 1 is proved as above, and the analogue of lemma 2 can be proved in the same way ((C2) is true, for $C = p$, since $L(\Gamma) < 0 = L(p)$ holds for no $\Gamma \in K$). The analogue of theorem 2 is also true. For axioms $\vdash p$ in Φ, we have $\Lambda \in \mu(p^o)$, by $\vdash_{\mathbf{S}} p^o$ and the definition of $\mu(p^o)$. As a consequence, we obtain fmp for all axiomatic extensions of $\mathbf{L1}[\wedge]$ by a finite number of axioms of the form $\vdash p$.

Further strengthenings of systems, considered above, can be defined by introducing *the exchange rule*:

$$\text{(E)}\ \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C}.$$

The system $\mathbf{L}$ + (E) is denoted by $\mathbf{LE}$, and similarly for other systems. In systems with (E), one proves $A \to B \vdash B \leftarrow A$ and $B \leftarrow A \vdash A \to B$, and consequently, $A \to B$ and $B \leftarrow A$ are logically equivalent; standard versions of these systems use $\to$ only, and the rules for $\leftarrow$ are omitted.

LE is the van Benthem calculus of semantic types [2, 3], and **L1E** equals the BCI-logic (it is also the implicational fragment of MALL and ILL; see Ono [14]).

Fmp can easily be proved for systems **LE** and **L1E** (also with $\wedge$) and their finite and bounded axiomatic extensions, using a slightly modified method of barriers. Most general algebraic frames for **LE** (resp. **L1E**) are commutative residuated semigroups (resp. monoids) (the operation $\circ$ is commutative; then, $x \Rightarrow y = y \Leftarrow x$, for all $x, y \in M$, hence one regards $\Rightarrow$ only). Powerset frames over commutative semigroups (resp. monoids) are commutative residuated semigroups (resp. monoids), and one interprets $\wedge$ as the intersection of sets. To construct a canonical model one uses finite multisets of formulas instead of strings of formulas. By $M_1(T)$ (resp. $M(T)$) we denote the set of all (resp. nonempty) finite multisets whose elements belong to T. Clearly, $(M(T), \cdot)$ (resp. $(M_1(T), \cdot, \lambda)$), where $\cdot$ stands for the join of multisets, and λ for the empty multiset, is a free commutative semigroup (resp. monoid). Using the methods described above, one proves that **LE** (also **LE**$[\wedge]$) is strongly complete with respect to powerset models over free commutative semigroups, and **L1E** (also **L1E**$[\wedge]$) is strongly complete with respect to powerset models over free commutative monoids.

Restricted powerset frames over commutative semigroups and monoids are defined as for arbitrary semigroups and monoids. The class $\mathcal{F}^c$ consists of restricted powerset frames $P_K(\mathcal{M})$ such that $\mathcal{M} = (M(T), \cdot)$, T is finite, and $K \subseteq M(T)$ is nonempty and finite and satisfies:

(MK') for all $x, y \in M(T)$, if $x \cdot y \in K$, then $x \in K$ and $y \in K$.

The class $\mathcal{F}^c_1$ consists of restricted powerset frames $P_K(\mathcal{M})$ such that $\mathcal{M} = (M_1(T), \cdot, \lambda)$, T is finite, and $K \subseteq M_1(T)$ is nonempty and finite and satisfies (MK') and $\lambda \in K$.

Barriers are defined as above except that the parts for $\leftarrow$ are redundant. The construction of model (K, μ) before lemma 2 is modified by substituting $M(T)$ for T^+ everywhere. Then, lemma 2 and theorems 1 and 2 can be proved for **LE**, **LE**$[\wedge]$ and their finite and bounded axiomatic extensions in precisely the same way (of course, $\mathcal{F}^c$-models take the part of $\mathcal{F}$-models). Analogously, for **L1E**, **L1E**$[\wedge]$ and their finite and bounded axiomatic extensions the completeness with respect to $\mathcal{F}^c_1$-models can be proved by an appropriate modification of the arguments for **L1**, **L1**$[\wedge]$ and their extensions (one uses $M_1(T)$ instead of T^* and λ instead of Λ). (See also the remarks after theorem 3 which concern axioms $\vdash p$.)

The proof of fmp for **L1E** by the method of barriers has been first given in [9], answering a question from Meyer and Ono [13], and here the method is slightly simplified and generalized to axiomatic extensions (for **LE** a different proof has been given in [6]). Actually, fmp for **L1E** follows from fmp for MALL, established in Lafont [11], since **L1E** is a conservative fragment of MALL. However, Lafont [11] uses finite phase space models instead of our $\mathcal{F}^c_1$-models. We show that Lafont's theorem can also be proved by our methods.

6 Phase space models

Recall that *a phase space* is defined as a pair $(\mathcal{M}, \perp^\bullet)$ such that $\mathcal{M} = (M, \cdot, 1)$ is a commutative monoid, and $\perp^\bullet \subseteq M$. For $X, Y \subseteq M$, one defines $X \circ Y$ and $X \Rightarrow Y$ as above, and one sets $X^\perp = X \Rightarrow \perp^\bullet$. The formulas of MALL are formed out of propositional variables $p, q, r, \ldots$, 'negated' propositional variables $p^\perp, q^\perp, r^\perp, \ldots$ and constants $0, 1, \perp, \top$ by means of binary operation symbols $\otimes, \oplus, \wedge, \vee$. We change the original notation of Girard [10] in the spirit of Troelstra [16]: $\otimes, \oplus$ are the multiplicative conjunction and disjunction, respectively, $\wedge, \vee$ are their additive companions, $1, 0$ represent neutral elements for $\otimes$ and $\oplus$, respectively, and $\perp, \top$ represent the bottom and top, respectively, in lattices interpreting the additive operations.

Let $(\mathcal{M}, \perp^\bullet)$ be a phase space. A set $X \subseteq M$ is called *a fact*, if $X^{\perp\perp} = X$ (equivalently, $X = Y^\perp$, for some $Y \subseteq M$). A phase space model is determined by an assignment μ of facts to propositional variables. One extends μ to all formulas by setting:

$$\mu(p^\perp) = \mu(p)^\perp,$$

$$\mu(1) = (\perp^\bullet)^\perp,\ \mu(0) = (\perp^\bullet)^{\perp\perp},\ \mu(\top) = M,\ \mu(\perp) = M^\perp,$$

$$\mu(A \otimes B) = (\mu(A) \circ \mu(B))^{\perp\perp},\ \mu(A \oplus B) = (\mu(A)^\perp \circ \mu(B)^\perp)^\perp,$$

$$\mu(A \wedge B) = \mu(A) \cap \mu(B),\ \mu(A \vee B) = (\mu(A)^\perp \cap \mu(B)^\perp)^\perp.$$

The formula A is said to be *true* in the phase space model $(\mathcal{M}, \perp^\bullet, \mu)$, if $1 \in \mu(A)$. Girard [10] proves the completeness of MALL with respect to phase space models: A is provable in MALL if, and only if, A is true in all phase space models.

Let us consider the system **L1E**$[\wedge, \top]$, i.e. **L1E**$[\wedge]$ enriched with $\top$ and axioms:

$$(\top)\ \Gamma \vdash \top,$$

for all sequences Γ, of formulas (with $\top$). In powerset models over monoids $(M, \cdot, 1)$ we interpret $\mu(\top) = M$, and in restricted powerset models with universe K we interpret $\mu(\top) = K$. An easy application of the methods described above yields the strong completeness of $\mathbf{L1E}[\wedge, \top]$ with respect to (restricted) powerset models over free commutative monoids.

We define a syntactic interpretation I of MALL in $\mathbf{L1E}[\wedge, \top]$. We fix a propositional variable s, supposed not to occur in formulas A in the domain of I, and we set:

$$I(p) = p \to s,\ I(p^{\perp}) = I(p) \to s,\ \text{for all variables } p \neq s,$$

$$I(1) = s \to s,\ I(0) = I(1) \to s,\ I(\top) = \top,\ I(\bot) = \top \to s,$$

$$I(A \otimes B) = [I(A) \to (I(B) \to s)] \to s,$$

$$I(A \oplus B) = (I(A) \to s) \to [(I(B) \to s) \to s],$$

$$I(A \wedge B) = I(A) \wedge I(B),$$

$$I(A \vee B) = (I(A) \to s) \wedge (I(B) \to s) \to s.$$

Lemma 3 *Let $(P(\mathcal{M}), \mu)$ be a powerset model over monoid $\mathcal{M} = (M, \cdot, 1)$. Let $(\mathcal{M}, \bot^{\bullet}, \mu')$ be the phase space model fulfilling: $\bot^{\bullet} = \mu(s)$, $\mu'(p) = \mu(p \to s)$, for all variables p. Then, for every formula A, of MALL, which does not contain s, $\mu'(A) = \mu(I(A))$.*

Proof. Induction on the complexity of A. For $\otimes$, one uses the equality $(X \circ Y) \Rightarrow Z = X \Rightarrow (Y \Rightarrow Z)$ which holds in powerset frames over commutative monoids. $\Box$

This lemma together with the completeness of $\mathbf{L1E}[\wedge, \top]$ with respect to powerset models and the soundness of MALL with respect to phase space models yields:

Corollary 1 *For every formula A, of MALL, not containing s, if A is provable in MALL, then $I(A)$ is provable in* $\mathbf{L1E}[\wedge, \top]$.

Lemma 4 *Let $(\mathcal{M}, \bot^{\bullet}, \mu)$ be a phase space model. Let $(P(\mathcal{M}), \mu')$ be the powerset model fulfilling: $\mu'(s) = \bot^{\bullet}$, $\mu'(p) = \mu(p) \Rightarrow \bot^{\bullet}$, for all variables $p \neq s$. Then, for every formula A, of MALL, not containing s, $\mu(A) = \mu'(I(A))$.*

PROOF. Induction on the complexity of A. Since $\mu(p)$ is a fact, we have:

$$\mu'(I(p)) = \mu'(p \to s) = \mu'(p) \Rightarrow \mu'(s) = \mu(p)^{\perp\perp} = \mu(p),$$

for every variable $p \neq s$, and the remaining clauses are straightforward. □.

Lemma 4 together with the completeness of MALL with respect to phase space models and the soundness of $\mathbf{L1E}[\wedge, \top]$ with respect to powerset models yields:

Corollary 2 *For every formula A, of MALL, not containing s, if $I(A)$ is provable in $\mathbf{L1E}[\wedge, \top]$, then A is provable in MALL.*

Accordingly, I is a faithful interpretation of MALL in $\mathbf{L1E}[\wedge, \top]$. The completeness of MALL with respect to finite phase space models easily follows from the next theorem.

Theorem 4 $\mathbf{L1E}[\wedge, \top]$ *is complete with respect to $\mathcal{F}_1^c$-models (K, μ) in which we interpret $\mu(\top) = K$.*

PROOF. Barriers for (Φ, Ψ), where Φ, Ψ are sets of simple sequents of $\mathbf{L1E}[\wedge, \top]$, are defined as for $\mathbf{L1E}[\wedge]$ with:

(B8) $L(\top) = 0$, if $\top \in F(\Phi, \Psi)$ is positive,

(B9) $L(\top) = 1$, if $\top \in F(\Phi, \Psi)$ is negative.

The value $U(\top)$ is inessential (but $L(A) \leq U(\top)$, for all $A \in F(\Phi, \Psi)$). The analogue of lemma 1 is proved nearly as above. One takes $U^{\sim}(\top) = 0$, if $\top \notin F(\Phi, \Psi)$; otherwise $U^{\sim}(\top) = U(\top)$. One takes $L^{\sim}(\top) = 0$, for positive $\top \in F(\Phi, \Psi')$, and $L^{\sim}(\top) = 1$, for negative $\top \in F(\Phi, \Psi')$.

The model (K, μ) is constructed as above with $\mu(\top) = K$. The analogue of lemma 2 is proved in the same way (observe that (C1)-(C4) hold for $C = \top$).

Obviously, $\mathbf{L1E}[\wedge, \top]$ is sound with respect to models of that kind. We prove completeness. Let $\vdash A$ be not derivable in $\mathbf{L1E}[\wedge, \top]$. We take $\Phi = \emptyset$, $\Psi = \{\vdash A\}$, and construct model (K, μ) as in the analogue of lemma 2. As above, we get $\lambda \notin \mu(A)$, hence A is not true in (K, μ). □

We give another proof of the result of Lafont [11].

Corollary 3 *MALL is complete with respect to finite phase space models.*

Proof. Soundness is obvious. To prove completeness assume A be not provable in MALL. By corollary 2, $I(A)$ is not provable in $\mathbf{L1E}[\wedge, \top]$ (we choose for s an arbitrary variable not occurring in A). By the above theorem, there exists an $\mathcal{F}_1^c$-model (K, μ) such that $\lambda \notin \mu(I(A))$. Clearly, the model (K, μ), i.e. the restricted powerset model $(P_K(\mathcal{M}), \mu)$ is isomorphic to a finite powerset model $(P(\mathcal{N}), \mu_1)$ such that $\mathcal{N} = (N, \odot, \lambda)$ is a finite commutative monoid, $N = K \cup \{a\}$, where a is a new element, and, for $x, y \in N$, we set:

$$x \odot y = x \cdot y, \text{ if } x \cdot y \in K; \quad \text{otherwise } x \odot y = a.$$

We define $\mu_1(p) = \mu(p) \cup \{a\}$, for all p occurring in $I(A)$. Then, $\mu_1(B) = \mu(B) \cup \{a\}$, for all subformulas B of $I(A)$. Consequently, $\lambda \notin \mu_1(I(A))$. By lemma 3, there is a finite phase space model $(\mathcal{N}, \perp^\bullet, \mu')$ such that $\lambda \notin \mu'(A)$. $\square$

Since $\mathbf{L1E}[\otimes, \wedge]$ is a conservative fragment of MALL, the above result yields the completeness of $\mathbf{L1E}[\otimes, \wedge]$ with respect to finite phase space models. Since the noncommutative version of theorem 4 is also true, one may expect that completeness theorems for noncommutative linear logics with respect to finite noncommutative phase space models can be proved in a similar way.

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