Cut elimination
for the Lambek calculus of adjoints

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Abstract

Lambek [3] proposes a new approach to type grammars which is based on the notion of a pregroup. A logic adequate for pregroups has been presented in the form of a rewriting system. This logic is given here two Gentzen style axiomatizations, and the cut elimination theorem for them is proven.

1 Pregroups and free pregroups

A pregroup is a structure $(G, \leq, \cdot, l, r, 1)$ such that $(G, \leq, \cdot, 1)$ is a partially ordered monoid (p.o. monoid), and l, r are unary operations on G, satisfying:

$$a^l a \leq 1 \leq aa^l \text{ and } aa^r \leq a^r a, \tag{1}$$

for all $a \in G$. The elements a^l and a^r are called *the left adjoint* and *the right adjoint*, respectively, of a. In every pregroup the following conditions are valid:

$$1^l = 1 = 1^r,\ (a^l)^r = a = (a^r)^l, \tag{2}$$

$$(ab)^l = b^l a^l,\ (ab)^r = b^r a^r, \tag{3}$$

$$a \leq b \text{ iff } b^l \leq a^l \text{ iff } b^r \leq a^r. \tag{4}$$

By $\mathbf{Z}$ we denote the set of integers. Let G be a pregroup. For $a \in G$ and $n \in \mathbf{Z}$, one defines an element $a^{(n)} \in G$, as follows:

$$a^{(0)} = a,\ a^{(n+1)} = (a^{(n)})^r,\ a^{(n-1)} = (a^{(n)})^l. \tag{5}$$

By (2), this definition is correct. Also, the following conditions are valid in pregroups, for all elements a, b and $n \in \mathbf{Z}$:

$$(ab)^{(2n)} = a^{(2n)} b^{(2n)},\ (ab)^{(2n+1)} = b^{(2n+1)} a^{(2n+1)}, \tag{6}$$

$$a \leq b \text{ iff } a^{(2n)} \leq b^{(2n)} \text{ iff } b^{(2n+1)} \leq a^{(2n+1)}. \tag{7}$$

A logic of pregroups can be presented in the form of a rewriting system, due to Lambek [3]. Let $(P, \leq)$ be a nonempty poset. *Atoms* are expressions of the form $p^{(n)}$, for $p \in P$, $n \in \mathbf{Z}$. We assume $p = p^{(0)}$. Finite strings of atoms are called *types* and denoted by X, Y, Z, U, V. Basic rewriting rules are the following:

(CON) Contraction: $X, p^{(n)}, p^{(n+1)}, Y \rightarrow X, Y$

(EXP) Expansion: $X, Y \rightarrow X, p^{(n+1)}, p^{(n)}, Y$

(IND) $X, p^{(2n)}, Y \rightarrow X, q^{(2n)}, Y$ and $X, q^{(2n+1)}, Y \rightarrow X, p^{(2n+1)}, Y$, if $p \leq q$ in $(P, \leq)$ (Induced Steps)

The relation $\Rightarrow$ is defined as the reflexive and transitive closure of the relation $\rightarrow$.

We define an equivalence relation: $X \sim Y$ iff both $X \Rightarrow Y$ and $Y \Rightarrow X$. It is nontrivial, since e.g. $p, p^{(1)}, p \sim p$. (Notice that $aa^l a = a$ and $aa^r a = a$ are valid in pregroups.) Formal adjoints of types are defined as follows: $\epsilon^l = \epsilon^r = \epsilon$ (ϵ denotes the empty string), and

$$(p^{(n)})^l = p^{(n-1)},\ (p^{(n)})^r = p^{(n+1)}, \tag{8}$$

$$(a_1 \ldots a_k)^l = a_k^l \ldots a_1^l,\ (a_1 \ldots a_k)^r = a_k^r \ldots a_1^r, \tag{9}$$

for all $p \in P$, $n \in \mathbf{Z}$, $k \geq 1$ and atoms $a_1, \ldots, a_k$. One easily shows that $X \sim Y$ entails $Z, X \sim Z, Y$, $X, Z \sim Y, Z$, $X^l \sim Y^l$ and $X^r \sim Y^r$. (First, show analogous conditions for $\Rightarrow$.) Then, on equivalence classes of $\sim$ one can define operations $\cdot, l, r$, an ordering $\leq$ and an element 1 as follows:

$$[X] \cdot [Y] = [X, Y],\ [X]^l = [X^l],\ [X]^r = [X^r], \tag{10}$$

$$[X] \leq [Y] \text{ iff } X \Rightarrow Y,\ 1 = [\epsilon]. \tag{11}$$

The structure defined above is a pregroup. It is called *the free pregroup generated by* the poset $(P, \leq)$.

Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets. A mapping $f : P_1 \mapsto P_2$ is said to be *order preserving*, if $a \leq_1 b$ entails $f(a) \leq_2 f(b)$, for all $a, b \in P_1$. It is easy to prove that, for any pregroup G, every order preserving mapping $h : P \mapsto G$ is uniquely extendable to a homomorphism of the free pregroup generated by $(P, \leq)$ into G. Further, $[X] \leq [Y]$ is true in the free pregroup if, and only if, $h([X]) \leq h([Y])$ is true for all homomorphisms h, obtained in this way (all homomorphisms, indeed). Since $[X] \leq [Y]$ is equivalent to $X \Rightarrow Y$, then the rewriting system described above yields the inequalities valid in pregroups.

Lambek [3] proves an important combinatorial property of this system. One introduces new rewriting rules: Generalized Contraction (GCON) and Generalized Expansion (GEXP) as follows:

(GCON) $X, p^{(n)}, q^{(n+1)}, Y \to X, Y$ if either n is even and $p \leq q$ in P, or n is odd and $q \leq p$ in P,

(GEXP) $X, Y \to X, p^{(n+1)}, q^{(n)}, Y$ if either n is even and $p \leq q$ in P, or n is odd and $q \leq p$ in P.

Clearly, (GCON) amounts to (IND) followed by (CON), and (GEXP) amounts to (EXP) followed by (IND). The Lambek Switching Lemma is: if $X \Rightarrow Y$, then there exist U, V such that $X \Rightarrow U$, $U \Rightarrow V$ and $V \Rightarrow Y$, and X can be transformed into U by (GCON) only, U into V by (IND) only, and V into Y by (GEXP) only. As a consequence, if $X \Rightarrow Y$ and Y is an atom or the empty string, then X can be transformed into Y by (CON) and (IND) only; also, if $X \Rightarrow Y$ and X is an atom or the empty string, then X can be transformed into Y by (EXP) and (IND) only. In [1], this fact is used to prove that grammars based on the calculus of free pregroups generate context-free languages; actually, they are weakly equivalent to context-free grammars (without $\epsilon-$productions). Another important consequence is the polynomial time decidability of $X \Rightarrow Y$. For $X \Rightarrow Y$ iff $Y^l, X \Rightarrow \epsilon$, and the latter must be derived by (CON) and (IND) only. These derivations can be simulated by a context-free grammar whose size is polynomially bounded by the size of the entry and the size of P, hence a standard parsing algorithm for context-free grammars yields a polynomial time decision procedure for this system.

In section 2 we present a Gentzen style sequent system equivalent to this rewriting system and prove cut elimination. Actually, we make it clear that the cut elimination theorem for the new system is essentially the same as the Switching Lemma for the rewriting system. In section 3 we present a one-side sequent system for compact bilinear logic (CBL) which is another formalization of the logic of pregroups and we prove cut elimination.

2 The calculus of adjoints

The rewriting system described in section 1 can also be presented as a Gentzen style sequent calculus. Let $(P, \leq)$ be fixed. Atoms and types are defined as above. *Sequents* are of the form $X \Rightarrow Y$, where X, Y are types. The axioms and the inference rules of the calculus are the following:

$$\text{(Id) } X \Rightarrow X$$

$$\text{(LA) } \frac{X, Y \Rightarrow Z}{X, p^{(n)}, p^{(n+1)}, Y \Rightarrow Z}, \quad \text{(RA) } \frac{X \Rightarrow Y, Z}{X \Rightarrow Y, p^{(n+1)}, p^{(n)}, Z}$$

$$\text{(LIND) } \frac{X, p^{(n)}, Y \Rightarrow Z}{X, q^{(n)}, Y \Rightarrow Z}, \quad \text{(RIND) } \frac{X \Rightarrow Y, p^{(n)}, Z}{X \Rightarrow Y, q^{(n)}, Z}$$

(LIND) is limited by: $q \leq p$ if n is even, and $p \leq q$ if n is odd. (RIND) is limited by: $p \leq q$ if n is even, and $q \leq p$ if n is odd. X, Y, Z are arbitrary types, p, q are arbitrary elements of P, and n is an arbitrary integer.

The cut rule is the following:

$$\text{(CUT)}\ \frac{X \Rightarrow Y;\ Y \Rightarrow Z}{X \Rightarrow Z}.$$

Let S denote the system axiomatized by (Id), (LA), (RA), (LIND) and (RIND), and let S' denote S enriched with (CUT).

Proposition 1 *For all types X, Y, $X \Rightarrow Y$ holds in the sense of the rewriting system if, and only if, $X \Rightarrow Y$ is provable in S'.*

Proof. Assume $X \Rightarrow Y$ hold in the sense of the rewriting system. Then, there exist types $Z_0, \ldots, Z_n$, $n \geq 0$, such that $Z_0 = X$, $Z_n = Y$, and $Z_{i-1} \rightarrow Z_i$, for all $1 \leq i \leq n$. We show that $Z_{i-1} \Rightarrow Z_i$ is provable in S', for all $1 \leq i \leq n$. If $Z_{i-1} \rightarrow Z_i$ is (CON), then we apply (LA) to axiom $Z_i \Rightarrow Z_i$. If $Z_{i-1} \rightarrow Z_i$ is (EXP), then we apply (RA) to axiom $Z_{i-1} \Rightarrow Z_{i-1}$. If $Z_{i-1} \rightarrow Z_i$ is (IND), then we apply (RIND) to axiom $Z_{i-1} \Rightarrow Z_{i-1}$; we can also apply (LIND) to axiom $Z_i \Rightarrow Z_i$. So, if $n = 0$, then $X \Rightarrow Y$ is axiom (Id), and if $n > 0$, then $X \Rightarrow Y$ is provable in S', using (CUT).

Assume $X \Rightarrow Y$ be provable in S'. We prove that $X \Rightarrow Y$ holds in the rewriting system, by induction on axioms and rules of S'. If $X \Rightarrow Y$ is (Id), then the claim is true. For inference rules, we show that if the premise (premises) holds (hold) in the rewriting system, then the conclusion holds in this system. For (LA), the antecedent of the conclusion can be transformed into the antecedent of the premise, by (CON). For (RA), the consequent of the premise can be transformed into the consequent of the conclusion, by (EXP). For (LIND), the antecedent of the conclusion can be transformed into the antecedent of the premise, by (IND). For (RIND), the consequent of the premise can be transformed into the consequent of the conclusion, by (IND). For (CUT), if the premises hold in the rewriting system, then the conclusion also holds in this system, since $\Rightarrow$ is transitive. $\Box$

It follows from this proof that S' is equivalent to S' without (LIND) (or: without (RIND)). We prove a form of cut elimination.

Theorem 1 *For all types X, Y, $X \Rightarrow Y$ is provable in S if, and only if, $X \Rightarrow Y$ is provable in S'.*

Proof. The 'only if' part is obvious. So, assume that $X \Rightarrow Y$ be provable in S'. By proposition 1, $X \Rightarrow Y$ holds in the rewriting system. By the Switching Lemma, there exist types U, V such that $X \Rightarrow U$ holds by (GCON) only, $U \Rightarrow V$ holds by (IND) only, and $V \Rightarrow Y$ holds by (EXP) only. Consequently, $X \Rightarrow U$ holds by (CON) and (IND) only, and $U \Rightarrow Y$ holds by (EXP) and (IND) only.

Then, there exist types $Z_0, \ldots, Z_m$, $m \geq 0$, such that $Z_0 = X$, $Z_m = U$, and $Z_{i-1} \to Z_i$ is either (CON), or (IND), for all $1 \leq i \leq m$. We show that $Z_i \Rightarrow U$ is provable in S, for all $0 \leq i \leq m$. $Z_m \Rightarrow U$ is axiom (Id). Assume $Z_i \Rightarrow U$ be provable in S, $i > 0$. If $Z_{i-1} \to Z_i$ is (CON), then $Z_{i-1} \Rightarrow U$ is the result of applying (LA) to $Z_i \Rightarrow U$. If $Z_{i-1} \to Z_i$ is (IND), then $Z_{i-1} \Rightarrow U$ is the result of applying (LIND) to $Z_i \Rightarrow U$. Consequently, $X \Rightarrow U$ is provable in S'.

Now, there exist types $V_0, \ldots, V_n$, $n \geq 0$, such that $V_0 = U$, $V_n = Y$, and $V_{i-1} \to V_i$ is either (EXP), or (IND), for all $1 \leq i \leq n$. We show that $X \Rightarrow V_i$ is provable in S, for all $0 \leq i \leq n$. $X \Rightarrow V_0$ is provable in S, by the first part of the proof. Assume $X \Rightarrow V_{i-1}$ be provable in S, $1 \leq i$. If $V_{i-1} \to V_i$ is (EXP), then $X \Rightarrow V_i$ is the result of applying (RA) to $X \Rightarrow V_{i-1}$. If $V_{i-1} \to V_i$ is (IND), then $X \Rightarrow V_i$ is the result of applying (RIND) to $X \Rightarrow V_{i-1}$. We have shown that $X \Rightarrow Y$ is provable in S. $\Box$

This proof suggests that both (LIND) and (RIND) are necessary in S. Indeed, if $p \leq q$ in P, then from axiom $\epsilon \Rightarrow \epsilon$ we infer $\epsilon \Rightarrow p^{(1)}, p$, by (RA), hence $\epsilon \Rightarrow p^{(1)}, q$, by (RIND). The latter sequent cannot be a conclusion of (LIND). It has exactly two proofs in S; the other one produces $\epsilon \Rightarrow q^{(1)}, q$, by (RA), hence $\epsilon \Rightarrow p^{(1)}.q$, by (RIND). Consequently, (RIND) is necessary in S. A dual argument shows that (LIND) is necessary. In S', this sequent can be proven, using (LIND): $p^{(1)}, p \Rightarrow p^{(1)}, q$ holds, by (LIND), and one uses (CUT) with the left premise $\epsilon \Rightarrow p^{(1)}, p$.

Of course, theorem 1 can also be proven directly, not using the Switching Lemma. One can show that (CUT) is admissible in S: if $X \Rightarrow Y$ and $Y \Rightarrow Z$ are provable in S, then $X \Rightarrow Z$ is so. The most easy way to prove this fact is similar to Lambek's proof of the Switching Lemma. We introduce rules (GLA) and (GRA). (GLA) arises from (LA) by replacing $p^{(n)}$ with $q^{(n)}$ and requiring: $q \leq p$ if n is even, and $p \leq q$ is n is odd. (GRA) arises from (RA) in the same way, but now we require: $p \leq q$ if n is even, and $q \leq p$ if n is odd. Clearly, (GLA) amounts to (LA) followed by (LIND), and (GRA) amounts to (RA) followed by (RIND). Accordingly, (GLA) and (GRA) are derivable in S, hence the system S'' axiomatized by (Id), (GLA), (GRA), (LIND) and (RIND) is equivalent to S. We prove that (CUT) is admissible in S''. We proceed by induction on proofs of $X \Rightarrow Y$ and $Y \Rightarrow Z$ in S''. The interesting cases are the following. (i) $X \Rightarrow Y$ follows from $X \Rightarrow Y', Y''$, by (GRA), with $Y = Y', p^{(n+1)}, q^{(n)}, Y''$, and $Y \Rightarrow Z$ is a result of applying (LIND) which touches one of the types $p^{(n+1)}, q^{(n)}$. If (LIND) touches $p^{(n+1)}$, then the premise of (LIND) is $Y', r^{(n+1)}, q^{(n)}, Y'' \Rightarrow Z$. Then, we change the instance of (GRA) to yield $X \Rightarrow Y', r^{(n+1)}, q^{(n)}, Y''$, drop the application of (LIND) and use the induction hypothesis. We only need to confirm the requirements to be fulfilled. If n is even, then $p \leq q$ and $r \leq p$, hence $r \leq q$, as required. If n is odd, then $q \leq p$ and $p \leq r$, hence $q \leq r$, as required. If (LIND) touches $q^{(n)}$, then we reason in a similar way. (ii) $X \Rightarrow Y$ follows from $X \Rightarrow Y', p^{(n)}, Y''$, by (RIND), with $Y = Y', q^{(n)}, Y''$, and $Y \Rightarrow Z$ is the result of applying (GLA) which touches $q^{(n)}$. Again, we can change the instance of (GLA), drop the application of (RIND), and use the induction hypothesis. (iii) $X \Rightarrow Y$ results from applying (RIND), and $Y \Rightarrow Z$ results from applying (LIND), and both applications of rules touch the same type in Y. Then, these two applications can be reduced to one application of (RIND). (iv) $X \Rightarrow Y$ results from applying (GRA), and $Y \Rightarrow Z$ results from applying (GLA), and the types introduced by these rules overlap. Say, the premise of (GRA) is $X \Rightarrow Y', Y''$, the premise of (GLA) is $U, U' \Rightarrow Z$, $Y = Y', p^{(n+1)}, q^{(n)}, Y''$, $U = Y', p^{(n+1)}$, $Y'' = r^{(n+1)}, U'$. Then, (RIND) applied to $X \Rightarrow Y', Y''$ yields $X \Rightarrow U', U''$, and we use the induction hypothesis. The other case is treated in a similar way.

The above proof is not much different from Lambek's proof of the Switching Lemma. So, the main message of this section is that the Switching Lemma is essentially the cut elimination theorem for the calculus of adjoints. It is not difficult to infer the Switching Lemma from the cut elimination theorem for S''. Thus, these two theorems are of the same power.

3 Compact bilinear logic

Pregroups can also be seen as models of compact bilinear logic (CBL) which arises from the multiplicative part of classical linear logic of Girard [2] by identifying 'times' with 'par'. The language of CBL is a propositional language with the constant 1, the binary connective $\otimes$, and unary connectives l, r. We present an axiomatization of CBL in the form of a one-side sequent system admitting cut elimination.

Atoms are 1 and types $p^{(n)}$ such that p is a variable and $n \in \mathbf{Z}$. Formulas are formed out from atoms by $\otimes$: if A, B are formulas, then $(A \otimes B)$ is a formula. Adjoints are defined in the metalanguage. For an atom A, A^l and A^r are defined by (8) with $1^l = 1^r = 1$. We also define:

$$(A \otimes B)^l = B^l \otimes A^l,\ (A \otimes B)^r = B^r \otimes A^r. \tag{12}$$

One proves $(A^l)^r = A = (A^r)^l$, by induction on A.

Sequents are finite sequences of formulas. Sequents are denoted by Γ, Δ, Φ. Given a pregroup G, any assignment of elements of G to variables can uniquely be extended to an assignment of elements of G to all formulas (we interpret $\otimes$ as $\cdot$). A formula A is true under assignment h if $1 \leq h(A)$ in G, and a sequent $A_1, \ldots, A_n$ is true if the formula $A_1 \otimes \cdots \otimes A_n$ is true. (The empty sequent is always true.) The system CBL described below yields all sequents true in all pregroups under all assignments.

The axiom is ϵ. The inference rules are the following:

$$(1\mathrm{I})\ \frac{\Gamma, \Delta}{\Gamma, 1, \Delta}$$

$$(\otimes\mathrm{I})\ \frac{\Gamma, A, B, \Delta}{\Gamma, A \otimes B, \Delta}$$

$$\text{(lI)}\ \frac{\Gamma, \Delta}{\Gamma, A, A^l, \Delta}$$

$$\text{(rI)}\ \frac{\Gamma, \Delta}{\Gamma, A^r, A, \Delta}$$

We restrict (lI) and (rI) to $A = p^{(n)}$. Then, (lI) and (rI) are the same rule. We write $\vdash \Gamma$ if Γ is provable in this system.

Lemma 1 *If* $\vdash \Gamma, \Delta$ *then* $\vdash \Gamma, A, A^l, \Delta$ *and* $\vdash \Gamma, A^r, A, \Delta$*, for every formula* A*.*

PROOF. Induction on A. For $A = p^{(n)}$, it holds, by (lI), (rI). For $A = 1$, it holds, by (1I). Let $A = B \otimes C$. Then, $A^l = C^l \otimes B^l$. Assume $\vdash \Gamma, \Delta$. Then, $\vdash \Gamma, B, B^l, \Delta$, hence $\vdash \Gamma, B, C, C^l, B^l, \Delta$, by the induction hypothesis, and we apply ($\otimes$I). For A^r, the argument is similar. $\Box$

Lemma 2 *If* $\vdash \Gamma, \Delta$ *and* $\vdash \Phi$*, then* $\vdash \Gamma, \Phi, \Delta$*.*

PROOF. Induction on the proof of Φ. $\Box$

Lemma 3 *If* $\vdash \Gamma, 1, \Delta$*, then* $\vdash \Gamma, \Delta$*.*

PROOF. Induction on the proof of $\Gamma, 1, \Delta$. $\Box$

Lemma 4 *If* $\vdash \Gamma, A \otimes B, \Delta$*, then* $\vdash \Gamma, A, B, \Delta$*.*

PROOF. Induction on the proof of $\Gamma, A \otimes B, \Delta$. $\Box$

Notice that the next lemma is not a reversal of rules (lI) and (rI). It is crucial for cut elimination.

Lemma 5 *If* $\vdash \Gamma, A^l, A, \Delta$ *then* $\vdash \Gamma, \Delta$*. If* $\vdash \Gamma, A, A^r, \Delta$ *then* $\vdash \Gamma, \Delta$*.*

PROOF. This is not so obvious, as the preceding lemmas. We proceed by induction on A. For $A = 1$, we use lemma 3. Let $A = p^{(n)}$. We switch on induction on the proof of $\vdash \Gamma, A^l, A, \Delta$. We only consider the nontrivial case: this sequent results by application of (lI) which introduces one of the formulas A^l, A. (i) The rule introduces A^l. Then, $\Gamma = \Gamma', p^{(n)}$ and the premise is $\Gamma', p^{(n)}, \Delta$. So, the premise equals Γ, Δ. (ii) The rule introduces A. Then, $\Delta = p^{(n-1)}, \Delta'$ and the premise is $\Gamma, p^{(n-1)}, \Delta'$. Again, the premise equals Γ, Δ. Let $A = B \otimes C$. Our assumption is $\vdash \Gamma, C^l \otimes B^l, B \otimes C, \Delta$, so we can use lemma 4 and the induction hypothesis. The second part of the lemma can be proven in a similar way. $\Box$

We prove the cut elimination theorem.

Theorem 2 *If* $\vdash \Gamma, A^l$ *and* $\vdash A, \Delta$*, then* $\vdash \Gamma, \Delta$*. If* $\vdash \Gamma, A$ *and* $\vdash A^r, \Delta$*, then* $\vdash \Gamma, \Delta$*.*

PROOF. Assume $\vdash \Gamma, A^l$ and $\vdash A, \Delta$. By lemma 2, we infer $\vdash \Gamma, A^l, A, \Delta$, and consequently, $\vdash \Gamma, \Delta$, by lemma 5. The second part is analogous. $\Box$

We define Γ^l and Γ^r by setting $\epsilon^l = \epsilon^r = \epsilon$ and:

$$(A_1, \dots, A_n)^l = A_n^l, \dots, A_1^l,\ (A_1, \dots, A_n)^r = A_n^r, \dots, A_1^r.$$

Lemma 6 $\vdash \Gamma^r, \Delta$ *if, and only if,* $\vdash \Delta, \Gamma^l$.

PROOF. Assume $\vdash \Gamma^r, \Delta$. By lemma 1, we get $\vdash \Gamma, \Gamma^l$, hence $\vdash \Gamma, \Gamma^r, \Delta, \Gamma^l$, by lemma 2. Then, $\vdash \Delta, \Gamma^l$, by lemma 5. The converse conditional can be proven in a similar way. $\Box$

We write $\vdash \Gamma \Rightarrow \Delta$ iff $\vdash \Gamma^r, \Delta$ (or equivalently: $\vdash \Delta, \Gamma^l$). We prove another form of cut elimination.

Theorem 3 *If* $\vdash \Gamma \Rightarrow \Delta$ *and* $\vdash \Delta \Rightarrow \Phi$, *then* $\vdash \Gamma \Rightarrow \Phi$.

PROOF. Assume $\vdash \Gamma \Rightarrow \Delta$ and $\vdash \Delta \Rightarrow \Phi$. Then, $\vdash \Gamma^r, \Delta$ and $\vdash \Delta^r, \Phi$. Theorem 2 yields $\vdash \Gamma^r, \Phi$, hence $\vdash \Gamma \Rightarrow \Phi$. $\Box$

Now, it is easy to prove that this system is complete with respect to the semantics of pregroups. Evidently, it is sound. The completeness can be proven by a Lindenbaum style construction. We define $A \sim B$ iff $\vdash A \Rightarrow B$ and $\vdash B \Rightarrow A$. By theorem 3, $\sim$ is an equivalence relation. Further, it is a congruence on the syntactic algebra of CBL. To show this fact assume $\vdash A \Rightarrow B$. Then, $\vdash A^r, B$, hence we get $\vdash A^r, C^r, C, B$, by (rI), and consequently, $\vdash (C \otimes A)^r, (C \otimes B)$, by ($\otimes$I) and the definition of adjoints. Thus, $\vdash C \otimes A \Rightarrow C \otimes B$. In a similar way one shows $\vdash A \otimes C \Rightarrow B \otimes C$. Also, $\vdash A^r, B$ yields $\vdash B, A^l$, by lemma 6, which is $\vdash (B^l)^r, A^l$, and consequently, $\vdash B^l \Rightarrow A^l$. In a similar way one shows $\vdash B^r \Rightarrow A^r$. On the equivalence classes of $\sim$ we define operations $\cdot, l, r$ and the ordering $\leq$ by setting:

$$[A] \cdot [B] = [A \otimes B],\ [A]^l = [A^l],\ [A]^r = [A^r],$$

$$[A] \leq [B] \text{ iff } \vdash A \Rightarrow B.$$

The quotient structure is a pregroup with [1] being the unit. We have $\vdash A, A^l$, by (lI), hence $\vdash 1, A, A^l$, by (1I), and $\vdash 1, A \otimes A^l$, by ($\otimes$I), which yields $[1] \leq [A] \cdot [A]^l$. The condition $[A]^l \cdot [A] \leq [1]$ means $\vdash A^l \otimes A \Rightarrow 1$, and the latter amounts to $\vdash A^r \otimes A, 1$ which is provable, by (rI), (1I) and ($\otimes$I). In a similar way one proves the inequalities for right adjoints, associativity and monotonicity. Accordingly, if $\vdash A$ does not hold, then $\vdash 1, A$ does not hold either, by lemma 3, hence $[1] \leq [A]$ is not true, and consequently, the formula A is not true in this pregroup under the assignment $h(p) = [p]$, for variables p.

This system can be modified to admit a fixed ordering on the set of variables. Now, the variables should be treated as propositional constants, being elements of a poset $(P, \leq)$. We add a new rule:

$$\text{(RO)}\ \frac{\Gamma, p^{(n)}, \Delta}{\Gamma, q^{(n)}, \Delta},$$

licensed by: $p \leq q$ if n is even, $q \leq p$ if n is odd. All results of this section can be proven for this extended system. Only the proof of lemma 5 requires some change. As in section 2, we generalize rule (lI) (for atoms A) to the following one:

$$\text{(GlI)}\ \frac{\Gamma, \Delta}{\Gamma, p^{(n+1)}, q^{(n)}, \Delta},$$

licensed as for (RO). Clearly, (GrI) would be the same rule. (GlI) amounts to (lI) followed by (RO), so the system CBL' which results from CBL by replacing (lI) (for atoms A) with (GlI) is equivalent to CBL. Accordingly, it is sufficient to prove lemma 5 for CBL', which is not difficult.

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