Relations between Sets of Time Points and Quasi-Linear Orderings*

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1 Introduction

The mathematical contents of this paper are quite elementary. We consider binary relations $R \subset P(U) \times P(U)$, i.e. relations between subsets of a universe U. We prove that relations of that kind which satisfy certain additiveness conditions can be characterized as those which admit some simple first-order definition involving two unary predicates and a binary predicate, interpreted as a so-called *quasi-linear* ordering on U. Thus, this topic is obviously connected to the traditional concern of JÓNSSON and TARSKI [10] with additive operators definable by means of binary relations, which is an algebraic origin for Kripke frames for modal logics [2]. Some recent generalization of the Jónsson and Tarski approach to n-ary operators definable by means of $n+1$-ary relations can be found in DUNN [4, 5]. Yet, our perspective is different from that of Jónsson, Tarski and Dunn, since we deal with relations instead of operators, which enables us to obtain stronger results in some respect.

The real motivation of this paper comes from the logic of time, and more precisely, the logical semantics of some binary sentential connectives, as e.g. *after*, *before*. Let us consider sentences:

- John left after Paul arrived,
- John left before Paul arrived.

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A reasonable semantics for these constructions interprets sentences as sets of time moments; the interpretation of *John leaves* is the set of all moments t such that this sentence is true in moment t. Then, connectives *after*, *before* etc. can be interpreted as binary operators which assign a set of time moments to each pair of sets of time moments: 'p after q' is interpreted as the set of time moments in which this compound sentence is true, and similarly for 'p before q'.

The above approach is admitted in LANDMAN [13] who considers the following exact definitions of pAq ('p after q') and pBq ('p before q') in the retrospective meaning:

(A) $(pAq)(t_0) \;\Leftrightarrow\; \exists x(x < t_0 \wedge p(x) \wedge \exists y(y < x \wedge q(y)))$,

(B) $(pBq)(t_0) \;\Leftrightarrow\; \exists x(x < t_0 \wedge p(x) \wedge \exists y(x < y < t_0 \wedge q(y)))$.

Clearly, $(pAq)(t_0)$ could be written $t_0 \models pAq$, and similarly for other cases, which would change the above formulas into some standard logical stipulations of truth conditions for the connectives in question.

According to (A) and (B), pAq is equivalent to qBp in all models in which $<$ is transitive, hence in all models with a linear stream of time. It has been observed by several authors, e.g. ANSCOMBE [1], HEINÄMÄKI [8], HINRICHS [9], PARTEE [14], that this analysis is incompatible with numerous linguistic evidences against the equivalence of pAq and qBp. We follow the discussion in SANCHEZ VALENZIA, VAN DER WOUDEN and ZWARTS [15], which is the source of our interest in this matter. First, *before*, in contrast to *after*, manifests certain aspects of a *negative* word. For instance, negative polarity items such as *anyone* and *ever* can appear in *before*-clauses but not in *after*-clauses:

The children left before anyone had arrived.

*The children arrived after anyone had left.

The boys died before they ever reached Nice.

*The boys died after they ever reached Nice.

A general linguistic principle, coming from LADUSAW [12], expects negative polarity items in downward monotone contexts only; thus, pBq and $q' \subset q$ should entail pBq', but that is not guaranteed by the formula (B) in which both p and q occur positively. Another evidence is related to *veridicality*. pAq is veridical in both arguments, that means, from the truth of pAq at

t_0 one infers the truth of p and q at some moment $t < t_0$. Informally, *John left after Paul came* entails both *John left* and *Paul came*. On the contrary, *before* is veridical only in the first but not in the second argument: from the third item above one cannot conclude that the boys ever reached Nice, and from *John left the college before he graduated* one cannot conclude that John graduated. Yet, $\exists y(y < t_0 \wedge q(y))$ follows from pBq according to definition (B).

To account for the negative aspects of *before* LANDMAN [13] proposes another retrospective definition of pBq:

$$\text{(NB)}\ (pBq)(t_0) \;\Leftrightarrow\; \exists x(x < t_0 \wedge p(x) \wedge \forall y(y < t_0 \wedge q(y) \rightarrow x < y)).$$

Now, q occurs negatively in the defining condition, which makes pBq downward monotone in q. As a result, negative polarity items are admissible in *before*-clauses, and pBq is nonveridical in q, which agrees with linguistic observations of Anscombe, Heinämäki and others.

A further analysis of negative aspects of *before* is given in SANCHEZ VALENZIA et al. [15]. The authors observe that pBq is (at least in some standard readings) anti-additive in the second argument: $pBq \vee q'$ is equivalent to $(pBq) \wedge (pBq')$, witness the examples (mine):

John interrupted the play before Paul put his car or (Paul put) his house.

John interrupted the play before Paul put his car, and John interrupted the play before Paul put his house.

Also, pBq is additive in the first argument: $p \vee p'Bq$ is equivalent to $(pBq) \vee (p'Bq)$. The latter completely agrees with the form of (NB), but the former requires additional assumptions on $<$. The main technical result of [15] is: pBq is anti-additive in q, if $<$ is a linear ordering. If $<$ is not linear, then (NB) does not yield the desired anti-additiveness. Consider the model given in Figure 1 (we assume the bottom-up direction of time arrows). Both $(pBq)(t_0)$ and $(pBr)(t_0)$ hold, but $(pBq \vee r)(t_0)$ does not hold.

The above considerations entitle us to qualify the phrase p *before* ... as a *regular negation* in the terminology of ZWARTS [16], since it satisfies the De Morgan law:

$$\text{(DM)}\ N(q \vee r) \;\Leftrightarrow\; Nq \wedge Nr.$$

The $\Rightarrow$-half of this law is equivalent to the downward monotonicity. Downward monotone expressions which are not anti-additive are called *minimal negations*; examples are *few trees*, *at most three trees*, because:

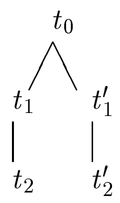

Figure 1: $p = \{t_2, t_2'\}$, $q = \{t_1\}$, $r = \{t_1'\}$

Few trees will blossom or will die $\Rightarrow$ Few trees will blossom and few trees will die,

At most three trees will blossom or will die $\Rightarrow$ At most three trees will blossom and at most three trees will die,

but the converse conditionals are not logically true. Examples of regular negations are *no trees*, *none of the trees*, since they make the biconditional logically true, and the phrase p *before* ... falls into this class. It seems to be of some philosophical interest that an ontological assumption about the structure of time (linearity) is needed to explain the linguistic behavior of *before* as an anti-additive expression.

The time is right to exhibit the major aim of this paper. We attack the following question: is (NB) the only possible definition for pBq which agrees with the linguistic evidences discussed above? And further: the results in [15] only show that the linearity of $<$ is a sufficient condition for the anti-additiveness of pBq; we would like to find a necessary and sufficient condition.

A bit surprisingly, we answer the first question in positive, but we need some additional property of pBq: irreflexivity. We also solve the second problem: the ordering relations must be quasi-linear, which is a generalization of linear orderings. Actually, these nice results can be obtained for finite universes (of time points) only, while infinite universes require some stronger assumptions (they also suggest certain essential refinements of the Zwarts hierarchy of negative expressions).

Our first step is to simplify the framework. In the setting proposed above, temporal connectives *after* and *before* are interpreted as binary operators on $P(U)$, where U is the universe of time points. We shall interpret them as binary relations on $P(U)$, i.e. sets $R \subset P(U) \times P(U)$. For $t \in U$, $p \subset U$, we define:

$$p_t = \{x \in p : x < t\}, \ p^t = \{x \in p : x > t\},$$

where $<$ is a fixed ordering relation on U. We also write $p(x)$ for $x \in p$. Then, if R is the interpretation of, say, *before*, then the retrospective reading of $(pBq)(t)$, that means, it is true at t that p happened before q happened, can be expressed by the formula:

$$(pBq)(t) \ \Leftrightarrow \ p_t R q_t, \tag{1}$$

while the prospective reading of $(pBq)(t)$, that means, it is true at t that p will happen before q will happen, can be expressed by the formula:

$$(pBq)(t) \ \Leftrightarrow \ p^t R q^t. \tag{2}$$

Thus, the formula:

(RNB) $pRq \ \Leftrightarrow \ \exists x(p(x) \wedge \forall y(q(y) \rightarrow x < y))$

defines a binary relation on $P(U)$. Clearly, if $(pBq)(t)$ is defined according to (1), then one obtains exactly the definition (NB). Similarly, (2) yields the prospective reading of $(pBq)(t)$ in Landman [13]:

$$(pBq)(t_0) \ \Leftrightarrow \ \exists x(x > t_0 \wedge p(x) \wedge \forall y(y > t_0 \wedge q(y) \rightarrow x < y)).$$

Now, our central question can be reformulated as follows: is (RNB) the only possible definition for the relation pBq which agrees with the linguistic evidences? To make this question precise we must list the conditions suggested by the discussion in [15]:

(C0) $pRq \ \Rightarrow \ p \neq \emptyset$ (veridicality in p),

(C0') $pRq \ \not\Rightarrow \ q \neq \emptyset$ (nonveridicality in q),

(C1) $(p \vee p')Rq \ \Leftrightarrow \ (pRq) \vee (p'Rq)$ (additiveness in p),

(C2) $pR(q \vee q') \ \Leftrightarrow \ (pRq) \wedge (pRq')$ (anti-additiveness in q).

One can easily show that (RNB) is not the only possible solution of conditions (C0)-(C2). For istance, the formula:

$$pRq \ \Leftrightarrow \ \exists x(p(x) \wedge \forall y(q(y) \rightarrow x \leq y)) \tag{3}$$

also yields a relation satisfying these conditions, while its linguistic sense corresponds to some reading of *not after*, as in:

John left not after Paul came,

rather than *before*. Thus, we add the new postulate:

(C3) $\neg(pRp)$ (irreflexivity),

obviously satisfied by *before* but not by *not after*.

In section 3 we prove that, in finite universes U, (RNB) is the only possible solution for conditions (C1), (C2) and (C3) with $<$ being a quasi-linear ordering (this notion is introduced in section 2). (Precisely, that holds for relations $R \subset P(U) \times P^+(U)$, where $P^+(U)$ denotes the family of nonempty subsets of U.) Infinite universes are considered in section 4; we obtain similar results for appropriately generalized conditions (C1), (C2) (complete additiveness instead of additiveness) and argue for refining the Zwarts hierarchy of negative expressions. In section 5 we discuss other relations of that kind which give rise to analogous characterizations.

2 Quasi-linear relations

Let $<$ be a binary relation on U. For $x \in U$, we set:

$$x^< = \{z \in U :\ x < z\},\ x^> = \{z \in U :\ z < x\},$$

that means, $x^<$ (resp. $x^>$) is the upper (resp. lower) principal cone of x with respect to $<$.

The relation $<$ on U is said to be *quasi-linear*, if it satisfies the conditions:

(IR) $\neg x < x$, for all $x \in U$ (irreflexivity),

(QC) $x^< \subset y^<$ or $y^< \subset x^<$, for all $x, y \in U$.

(QC) expresses quasi-connectedness.

Fact 1 *Quasi-linear relations are transitive.*

PROOF. Fix a quasi-linear relation $<$. Assume $x < y$ and $y < z$. Then, $y \in x^<$ but $y \notin y^<$, by (IR). Consequently, $x^< \not\subset y^<$, which yields $y^< \subset x^<$, by (QC). Since $z \in y^<$, then $z \in x^<$, which means $x < z$. $\Box$

Accordingly, quasi-linear relations are (strict) partial orderings, additionally satisfying (QC). Each linear ordering is quasi-linear. Figure 2 yields some examples of quasi-linear orderings which are not linear, while Figure 3 exhibits some partial orderings which are not quasi-linear (another example is given in Figure 1).

Roughly, quasi-linear orderings admit isolated points and branching of a sharply restricted form: if a node splits in two incompatible nodes, then

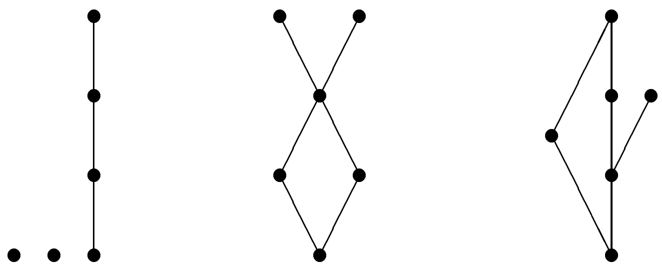

Figure 2: Quasi-linear orderings

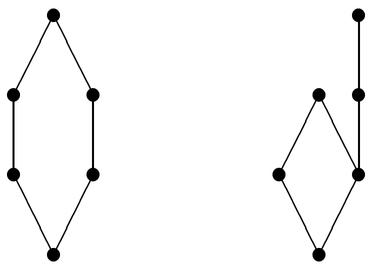

Figure 3: Non-quasi-linear partial orderings

one of them must either be an end-node (a leave), or join the upper cone of the other in a single step. So, there is essentially one main stream of the ordering.

It would be tempting to speculate about the intuitions behind quasi-linear relations as models of time, but we resigned from doing that here. This notion has simply been enforced by our characterization results. Accordingly, quasi-linearity should be treated as a notion needed to characterize certain set-relations, satisfying additiveness and reflexivity conditions.

In section 5, we use the following fact ($>$ denotes the converse of $<$).

Fact 2 *If $<$ satisfies (QC), then $>$ satisfies (QC). Consequently, if $<$ is quasi-linear, then $>$ is quasi-linear.*

Proof. Assume (QC) for $<$. Suppose that, for some $x, y \in U$, both $x^{>} \not\subset y^{>}$ and $y^{>} \not\subset x^{>}$. Then, there are $z < x$ and $u < y$ such that $z \not< y$ and $u \not< x$. So, $x \in z^{<}$ but $x \notin u^{<}$, and $y \in u^{<}$ but $y \notin z^{<}$, hence $z^{<} \not\subset u^{<}$ and $u^{<} \not\subset z^{<}$, which contradicts the assumption. $\Box$

3 Relations on finite universes

For $R \subset P(U) \times P(U)$, we define a relation $<_R \subset U \times U$, by setting:

$$x <_R y \;\Leftrightarrow\; \{x\}R\{y\}, \text{ for } x, y \in U.$$

Below we state the first characterization theorem for relations between subsets of U and nonempty subsets of U with U finite.

Theorem 1 *Let U be a finite set, and let $R \subset P(U) \times P^{+}(U)$. Then, R satisfies (C1), (C2) and (C3) if, and only if, R is defined by (RNB), for $p, q \subset U$, $q \neq \emptyset$, with a quasi-linear relation $<$.*

Proof. We prove ($\Rightarrow$). Assume that R fulfills (C1), (C2) and (C3). We show (RNB) with $< = <_R$.

Case I. $p = \emptyset$. The right hand side of (RNB) is false. If pRq, then qRq, by (C1), which contradicts (C3). So, the left hand side of (RNB) is also false.

Case II. $p \neq \emptyset$. Let:

$$p = \{x_1, \ldots, x_m\},\; q = \{y_1, \ldots, y_n\}.$$

By (C1), we obtain:

$$pRq \;\Leftrightarrow\; \{x_1\}Rq \vee \ldots \vee \{x_m\}Rq,$$

which means:

$$pRq \Leftrightarrow \exists x(p(x) \wedge \{x\}Rq).$$

By (C2), we obtain:

$$\{x\}Rq \Leftrightarrow \{x\}R\{y_1\} \wedge \ldots \wedge \{x\}R\{y_n\},$$

which means:

$$\{x\}Rq \Leftrightarrow \forall y(q(y) \rightarrow \{x\}R\{y\}).$$

(RNB) with $<=<_R$ results from joining the above formulas.

We show that $<$ is quasi-linear. (IR) obviously follows from (C3). We prove (QC). Let $x, y \in U$. We may assume $x^< \neq \emptyset$ and $y^< \neq \emptyset$ (otherwise, one of the conditions $x^< \subset y^<$, $y^< \subset x^<$ certainly holds). Define:

$$p = \{x, y\},\ q = x^<,\ r = y^<.$$

By (RNB), both pRq and pRr hold, which yields $pRq \vee r$, by (C2). By (RNB) again, we obtain:

$$x^< \cup y^< \subset x^< \text{ or } x^< \cup y^< \subset y^<,$$

and the latter is equivalent to:

$$y^< \subset x^< \text{ or } x^< \subset y^<.$$

We prove ($\Leftarrow$). Assume R is defined by (RNB) with $<$ quasi-linear. (C3) is a straightforward consequence of (RNB) and (IR). (C1) holds by easy logical calculations. We show (C2).

Assume $pRq \vee r$. By (RNB), we obtain:

$$\exists x(p(x) \wedge \forall y(q(y) \vee r(y) \rightarrow x < y)),$$

hence, by logic:

$$\exists x(p(x) \wedge \forall y(q(y) \rightarrow x < y) \wedge \forall y(r(y) \rightarrow x < y)),$$

and further:

$$\exists x(p(x) \wedge \forall y(q(y) \rightarrow x < y)) \wedge \exists x(p(x) \wedge \forall y(r(y) \rightarrow x < y)),$$

which yields pRq and pRr.

Now assume pRq and pRr. Then, by (RNB), there are $x_1, x_2 \in U$ such that:

$$p(x_1) \wedge \forall y(q(y) \rightarrow x_1 < y),$$

$$p(x_2) \wedge \forall y(r(y) \to x_2 < y).$$

Since $<$ is quasi-linear, we get $x_1^< \subset x_2^<$ or $x_2^< \subset x_1^<$. The former inclusion yields:

$$p(x_2) \wedge \forall y(q(y) \vee r(y) \to x_2 < y),$$

and the latter inclusion yields:

$$p(x_1) \wedge \forall y(q(y) \vee r(y) \to x_1 < y).$$

In both cases, $pRq \vee r$ holds, by (RNB). □

The restriction of the second domain of R to $P^+(U)$ is caused by the fact that the right hand side of (RNB) always holds, for $p \neq \emptyset$, $q = \emptyset$. On the other hand, conditions (C1)-(C3) do not entail $pR\emptyset$, for $p \neq \emptyset$. By (C2), $pR\emptyset$ holds, if pRq holds, for some q, but there may be nonempty sets p which do not satisfy pRq, for any q. To exclude the latter possibility we introduce the postulate:

(C4) $pR\emptyset$, for all $p \neq \emptyset$.

Clearly, (C2) and (C4) express together the finite anti-additiveness of R with respect to the second argument:

$$pR(q_1 \vee \ldots \vee q_m) \;\Leftrightarrow\; pRq_1 \wedge \ldots \wedge pRq_m,$$

for all $m \geq 0$, with $p \neq \emptyset$, if $m = 0$ (the empty disjunction equals 0, and the empty conjunction equals 1).

Theorem 2 *Let U be a finite set, and let $R \subset P(U) \times P(U)$. Then, R satisfies (C1)-(C4) if, and only if, R is defined by (RNB), for $p, q \subset U$, with a quasi-linear relation $<$.*

PROOF. The proof is similar to that of theorem 1. Only for Case II, one must consider the case $q = \emptyset$. Then, the left hand side of (RNB) holds, by (C4), and the right hand side holds, by logic. □

There is a one-one correspondence between relations $R \subset P(U) \times P(U)$, satisfying (C1)-(C4), and quasi-linear relations on U. For each quasi-linear relation $<$ on U determines a relation $R_<$, by (RNB), and each relation R, satisfying (C1)-(C4), determines a quasi-linear relation $<_R$, and these transformations are mutually converse. Linear orderings are precisely the maximal quasi-linear relations, hence we get a one-one correspondence between linear orderings on U and maximal relations on $P(U)$, satisfying (C1)-(C4).

Theorem 3 *Let U be a finite set, and let $R \subset P(U) \times P(U)$. Then, R is a maximal relation, satisfying (C1)-C4), if, and only if, R is defined by (RNB) with a linear ordering $<$.*

PROOF. Since $<$ is positive in (RNB), then the mapping which assigns $R_<$ to $<$ is upward monotone in $<$. Consequently, if $R_<$ is maximal satisfying (C1)-(C4), then $<$ is linear. Conversely, if $R_<$ is not maximal, then $R_< \subset R$, for some $R \neq R_<$, satisfying (C1)-C4). We have $< \subset <_R$ and $< \neq <_R$, since R is uniquely determined by $<_R$. Consequently, $<$ is not linear. □

Finally, we note that each relation R on $P(U)$ (U finite), satisfying (C1)-(C4), fulfills (C0) and is transitive:

$$\text{for all } p, q, r \subset U, \text{ if } pRq \text{ and } qRr, \text{ then } pRr,$$

hence it is a partial ordering on $P(U)$. In general, these relations are not quasi-linear.

4 Relations on infinite universes

For infinite universes U, conditions (C1)-(C4) are too weak to characterize the relations definable by (RNB). Let $\exists^\infty$ mean 'there are infinitely many'. Consider the relation R on $P(U)$, defined as follows:

$$pRq \iff (p \neq \emptyset \wedge q = \emptyset) \vee \exists^\infty x(p(x) \wedge \forall y(q(y) \to x < y)),$$

where $<$ is a linear ordering on U. R does not satisfy (RNB), since the relation $<_R$ is empty, and consequently, (RNB) defines the relation $P^+(U) \times \{\emptyset\}$, different from R. On the other hand:

Fact 3 *R fulfills (C1)-(C4).*

PROOF. We prove (C1). Assume $p \vee qRr$. If $r = \emptyset$, then $p \vee q \neq \emptyset$, hence $p \neq \emptyset$ or $q \neq \emptyset$, and we infer pRr or qRr. Let $r \neq \emptyset$. Then, there are infinitely many x such that:

$$(p(x) \vee q(x)) \wedge \forall y(r(y) \to x < y).$$

Consequently, one of the following sets:

$$\{x : p(x) \wedge \forall y(r(y) \to x < y)\},\ \{x : q(x) \wedge \forall y(r(y) \to x < y)\}$$

must be infinite, which yields pRr or qRr. The converse implication follows from the fact that the definition of R is upward monotone in the first argument.

We prove (C2). Assume $pRq \vee r$. If $q \vee r = \emptyset$, then $p \neq \emptyset$ and $q = \emptyset$, $r = \emptyset$, which yields pRq and pRr. Let $q \vee r \neq \emptyset$. Then, there are infinitely many x such that:

$$p(x) \wedge \forall y(q(y) \vee r(y) \rightarrow x < y),$$

which is equivalent to:

$$(p(x) \wedge \forall y(q(y) \rightarrow x < y)) \wedge (p(x) \wedge \forall y(r(y) \rightarrow x < y)).$$

Consequently, both pRq and pRr hold. Now, assume pRq and pRr. We consider several cases.

Case I. $q = r = \emptyset$. Then, $p \neq \emptyset$ and $q \vee r = \emptyset$, which yields $pRq \vee r$.

Case II. $q \neq \emptyset$, $r \neq \emptyset$. Then, either of the following sets:

$$A = \{x : p(x) \wedge \forall y(q(y) \rightarrow x < y)\},\ B = \{x : p(x) \wedge \forall y(r(y) \rightarrow x < y)\}$$

is infinite. We define:

$$C = \{x : p(x) \wedge \forall y(q(y) \vee r(y) \rightarrow x < y)\}.$$

If there is $a \in A$ such that $b \leq a$, for all $b \in B$, then $B \subset C$, hence C is infinite. If there is $b \in B$ such that $a \leq b$, for all $a \in A$, then $A \subset C$, and again C is infinite. Otherwise, there hold:

$$(\forall a \in A)(\exists b \in B)a < b \text{ and } (\forall b \in B)(\exists a \in A)b < a.$$

Consequently, $A = B = C$, hence C is infinite. Accordingly, $pRq \vee r$ holds.

Case III. $q \neq \emptyset$, $r = \emptyset$. With the notation above, A is infinite, and $A = C$, hence $pRq \vee r$ holds.

Case IV. $q = \emptyset$, $r \neq \emptyset$. Now, B is infinite, and $B = C$, which yields $pRq \vee r$.

(C3) and (C4) are easy consequences of the definition of R. $\square$

It follows from the above example that, for infinite universes, we need some strengthenings of conditions (C1)-(C4). Precisely, we should replace additiveness by complete additiveness, and a similar change should be done for anti-additiveness. The new conditions are as follows:

(IC1) $(\bigvee_{i \in I} p_i)Rq \;\Leftrightarrow\; \bigvee_{i \in I}(p_i Rq)$, (IC2) if $p \neq \emptyset$ is finite, then $pR(\bigvee_{i \in I} q_i) \;\Leftrightarrow\; \bigwedge_{i \in I}(pRq_i)$,

for all indexed families $\{p_i\}_{i \in I}$, $\{q_i\}_{i \in I}$ of subsets of U and all $p, q \subset U$, $x \in U$. The first argument of R in (IC2) is restricted to nonempty finite sets, since relations defined by (RNB) fulfill this restricted version of anti-additiveness, but they need not fulfill the full version (with p arbitrary). Consider the structure $(\mathcal{Z}, <)$ of integers with the natural ordering. Let $p = \mathcal{Z}$ and $q_i = \{i\}$, for any $i \in \mathcal{Z}$. Then, $pR_< q_i$ holds, for all $i \in \mathcal{Z}$, while $pR_<(\bigvee_{i \in I} q_i)$ does not hold.

We state the characterization theorem for arbitrary universes.

Theorem 4 *Let $R \subset P(U) \times P(U)$. Then, R satisfies (IC1), (IC2) and (C3) if, and only if, R is defined by (RNB) with a quasi-linear relation $<$.*

Proof. We prove $(\Rightarrow)$. Assume that R satisfies (IC1), (IC2), and (C3). For any set $p \subset U$, we have:

$$p = \bigvee_{x \in p} \{x\}.$$

Accordingly, pRq is equivalent to:

$$\bigvee_{x \in p} \{x\} Rq, \text{ by (IC1)},$$

$$\bigvee_{x \in p} \bigwedge_{y \in q} \{x\} R \{y\}, \text{ by (IC2) },$$

$$\exists x (p(x) \wedge \forall y (q(y) \to x <_R y)),$$

and consequently, R is defined by (RNB) with $< = <_R$. We show that $<$ is quasi-linear. (IR) follows from (C3). (QC) can be obtained in the same way as in the proof of theorem 1, using (IC2) instead of (C2).

To prove $(\Leftarrow)$ assume R be defined by (RNB) with a quasi-linear relation $<$. (C3) follows from (IR). (IC1) can be proven by easy logical calculations. The left-to-right direction of the equivalence in (IC2) holds, for all $p \subset U$, again by straightforward calculations. For the converse direction, assume pRq_i, for all $i \in I$, where $p \neq \emptyset$ is finite. If $I = \emptyset$, then $\bigvee_{i \in I} q_i = \emptyset$, and the left hand side of the equivalence holds, by (RNB). Let $I \neq \emptyset$. Then, for each $i \in I$, we pick out $x_i \in U$ such that:

$$p(x_i) \wedge \forall y (q_i(y) \to x_i < y).$$

Since p is finite, the indexed family $\{x_i\}_{i\in I}$ has finitely many different elements. So, the family $\{x_i^<\}_{i\in I}$ is essentially a finite chain of cones, since $<$ is quasi-linear, and consequently, there is a cone $x_j^<$ which contains all cones $x_i^<$, for $i \in I$. Clearly, there holds:

$$p(x_j) \wedge \forall y((\exists i \in I) q_i(y) \rightarrow x_j < y),$$

which yields the left hand side of the equivalence in (IC2). □

In the same way as theorem 3, we prove:

Theorem 5 *Let $R \subset P(U) \times P(U)$. Then, R is a maximal relation, satisfying (IC1), (IC2) and (C3) if, and only if, R is defined by (RNB) with a linear ordering $<$.*

Other conditions, considered in sections 3 and 4, are consequences of (IC1), (IC2) and (C3).

Fact 4 *Each relation R on U, satisfying (IC1), (IC2) and (C3), is transitive and fulfills (C0), (C2) and (C4).*

Proof. It follows from theorem 4 that each relation R, satisfying (IC1), (IC2) and (C3) is defined by (RNB) with a quasi-linear relation $<$. The transitivity of R is a consequence of (RNB) and the transitivity of $<$. (C0) follows from the form of (RNB). (C2) is proven in the same way as in the proof of theorem 1. To prove (C4) we use (IC1) and (IC2). By (IC2) with $I = \emptyset$, we obtain $\{x\}R\emptyset$, for all $x \in U$ (the empty conjunction equals the truth value 1). By (IC1), R is upward monotone with respect to the first argument, which yields (C4). □

Again, there is a one-one correspondence between relations on $P(U)$, satisfying (IC1), (IC2) and (C3), and quasi-linear relations on U, as well, as between maximal relations on $P(U)$, satisfying (IC1), (IC2) and (C3), and linear orderings on U.

Results of sections 3 and 4 fully explain the role of the formula (RNB) in defining relations, fulfilling linguistic requirements for the connective *before*. We have refined the key observation in Sanchez Valenzia et al. [15] that the linearity of $<$ is needed for the anti-additiveness of R in the second argument. We have shown that the exact condition required for anti-additiveness is quasi-linearity. It seems philosophically attractive to see that linguistic evidences for the additiveness properties of the connective *before* enforce a rather strong constraint on the structure of time, and it would be interesting to study other applications of quasi-linear orderings in temporal logics.

At the end of this section, we note that infinite universes suggest a need for refining the Zwarts hierarchy of negative expressions [16]. Recall that *minimal negations* are simply the downward monotone expressions; equivalently, they satisfy one half of (DM):

$$N(q \vee r) \;\Rightarrow\; N(q) \wedge N(r).$$

Examples of minimal negations are *few trees*, *at most two boys*. The expressions, satisfying (DM), are called *regular negations*; among such expressions, we find negative Noun Phrases, as e.g. *no boy*, *none of the trees*, and the phrase '*p before* ...'. Finally, *strong negations* satisfy both (DM) and the dual law:

$$\text{(DM')}\ N(q \wedge r) \;\Leftrightarrow\; N(q) \vee N(r).$$

In this class we find the sentential negation *it is not the case that*, and other linguistic representatives of the classical negation, as e.g. the negation of a Verb Phrase, the negation of a Determiner, and so on. Zwarts argues that certain negative polarity items can only appear in a regular (strong) negative context, but not in a minimal (regular) negative context. For instance, *need* can appear with a minimal negation, witness:

At most one woman need justify herself,

while *anything* requires a regular negation, witness:

*At most two students understood anything,

None of the students understood anything.

We claim that expressions qualified by Zwarts to regular negations are in fact completely anti-additive; they satisfy the following strenthening of (DM):

$$\text{(CDM)}\ N(\bigvee_{i \in I} q_i) \;\Leftrightarrow\; \bigwedge_{i \in I} N(q_i).$$

For instance, the sentence *no negative integer equals a positive integer* is equivalent to the infinite conjunction of sentences *no negative integer equals* n, for $n = 1, 2, 3, \ldots$. On the other hand, there are anti-additive expressions which are not completely anti-additive, witness *finitely many numbers*, *countably many numbers*, to be read: 'at most finitely many', 'at most countably many'. For the following schemes:

Finitely many natural numbers are P or Q,

Finitely many natural numbers are P, and finitely many natural numbers are Q,

are equivalent, while the next schemes:

Finitely many natural numbers are $\bigvee_{n \in N} P_n$,

$\bigwedge_{n \in N}$ finitely many natural numbers are P_n,

are nonequivalent. With P_n meaning 'is less than n', the former is false, and the latter is true. Accordingly, we find an intermediate class between Zwarts' minimal and regular negations: expressions satisfying (DM) but not (CDM), with *finitely many numbers*, *countably many numbers*, etc. as their representatives. We may call such expressions *weak regular negations*. Interestingly, weak regular negations do not admit negative polarity items of the strong type, like *any*, *ever*, witness the examples:

None of these transcendental numbers have any name,

* Finitely many transcendental numbers have any name,

None of these transcendental numbers have ever been applied,

* Finitely many transcendental numbers have ever been applied.

As a result, the linguistic behavior of weak regular negations is similar to that of minimal negations rather than regular negations. So, (CDM) rather than (DM) is a proper condition which characterizes the Zwarts class of regular negations.

Let us return to the connective *before*. The proof of theorem 4 yields, actually, the following:

Fact 5 *Each relation R on $P(U)$, satisfying (IC1), (IC2) and (C3), also satisfies the equivalence in (IC2), for all $p \subset U$ such that, for some $x \in p$, the cone $x^<$ contains all cones $y^<$, for $y \in p$.*

The condition of fact 5 is fulfilled, if there exists the $<$-least element in p. Thus, for all events p which have a beginning in time, the phrase p *before* $\ldots$ satisfies (CDM), hence it behaves as a regular negation in the above sense. That agrees with the presence of strong negative polarity items in *before*-clauses:

John left the city, before anything happened,

The theorem was published, before it found any application.

5 Related questions

In this section we discuss other relations on $P(U)$, satisfying certain additiveness and reflexivity conditions, and their connections with quasi-linear orderings.

We shall consider relations on $P^+(U)$ to avoid complications with empty arguments. For $R \subset P^+(U) \times P^+(U)$, we define *the converse* $R^\smile$ and *the complement* R^c, by setting:

$$pR^\smile q \;\Leftrightarrow\; qRp,$$

$$pR^c q \;\Leftrightarrow\; \neg pRq,$$

for $p, q \in P^+(U)$.

For the finite universe U, relations $R \subset P^+(U) \times P^+(U)$, satisfying (C1), (C2) and (C3), will be called *B-relations.* Relations of the form $R^\smile$ (resp. R^c, $R^{\smile c}$), where R is a B-relation, will be called $B^\smile$*-relations* (resp. B^c*-relations*, $B^{\smile c}$*-relations*). To characterize these kinds of relations we need duals of conditions (C1)-(C3):

(C1') $pR(q \vee q') \;\Leftrightarrow\; (pRq) \vee (pRq')$,

(C2') $(p \vee p')Rq \;\Leftrightarrow\; (pRq) \wedge (p'Rq)$,

(C3') pRp,

for all $p, p', q, q' \in P^+(U)$. There hold the following facts:

(A) R satisfies (C1) if, and only if, $R^\smile$ satisfies (C1'),

(B) R satisfies (C1) if, and only if, R^c satisfies (C2'),

(C) R satisfies (C2) if, and only if, $R^\smile$ satisfies (C2'),

(D) R satisfies (C2) if, and only if, R^c satisfies (C1'),

(E) R satisfies (C3) if, and only if, $R^\smile$ satisfies (C3),

(F) R satisfies (C3) if, and only if, R^c satisfies (C3').

Thus, $B^\smile$-relations can be defined by conditions (C1'), (C2') and (C3), B^c-relations by conditions (C1'), (C2') and (C3'), and $B^{\smile c}$-relations by conditions (C1), (C2) and (C3'). So, two possible combinations of additiveness and antiadditiveness mixed with two possible types of reflexivity yield four basic kinds of relations on $P^+(U)$:

(B) additive in p, anti-additive in q, irreflexive,

($B^{\smile}$) anti-additive in p, additive in q, irreflexive,

(B^{c}) anti-additive in p, additive in q, reflexive,

($B^{\smile c}$) additive in p, anti-additive in q, reflexive.

It follows from theorem 1 that B-relations R on $P^{+}(U)$ are precisely those which are defined by (RNB) with a quasi-linear relation $<$. Clearly, analogous characterizations can be obtained for the remaining kinds of relations.

$B^{\smile}$-relations are precisely those which can be defined by the formula:

$$(\mathrm{RNB}^{\smile})\ pRq \ \Leftrightarrow\ \exists x(q(x) \wedge \forall y(p(y) \rightarrow x < y)),$$

for a quasi-linear relation $<$. A linguistic representative of this kind is the phrase *before p, q*, as in *before Paul came, John left.*

B^{c}-relations are precisely those which can be defined by the formula:

$$(\mathrm{RNB}^{c})\ pRq \ \Leftrightarrow\ \forall x(p(x) \rightarrow \exists y(q(y) \wedge x \not< y))$$

with $<$ quasi-linear. As a linguistic representative, one may regard the phrase *it is not the case that p before q*.

$B^{\smile c}$-relations are precisely those which can be defined by the formula:

$$(\mathrm{RNB}^{\smile c})\ pRq \ \Leftrightarrow\ \forall x(q(x) \rightarrow \exists y(p(y) \wedge x \not< y))$$

with $<$ quasi-linear. The phrase *it is not the case that before p, q* is a linguistic representative of this kind. Notice that $B^{c\smile}$-relations are the same as $B^{\smile c}$-relations, since $R^{\smile c} = R^{c\smile}$, for every relation R.

For arbitrary universes U and relations $R \subset P^{+}(U) \times P^{+}(U)$, analogous results can be obtained with conditions (IC1), (IC2), restricted to nonempty sets I and $p, q, p_i, q_i \subset U$ (the constraint $p \neq \emptyset$ in (IC2) is redundant), and appropriate dual conditions. The latter are less elegant, however, because (IC1) leads to:

(IC1') $pR(\bigvee_{i\in I} q_i) \ \Leftrightarrow\ \bigvee_{i\in I}(pRq_i)$,

(IC2") $(\bigvee_{i\in I} p_i)Rq \ \Leftrightarrow\ \bigwedge_{i\in I}(p_iRq)$,

by taking the converse and the complement, respectively, while (IC2) leads to:

(IC2') equal to (IC2") for finite q,

(IC1") equal to (IC1) for finite p.

Unfortunately, (IC1') is not equivalent to (IC1"), and similarly for (IC2') and (IC2"). As above, we obtain four kinds of relation:

(B) conditions: (IC1), (IC2), (C3); definition: (RNB),

($B^{\smile}$) conditions: (IC1'), (IC2"), (C3); definition: (RNB$^{\smile}$),

(B^c) conditions: (IC1"), (IC2'), (C3'); definition: (RNB^c),

(B$^{\smile c}$) conditions: (IC1"), (IC2$^+$), (C3'); definition: (RNB$^{\smile c}$),

where (IC2$^+$) is (IC2) for arbitrary p. In each case, the relation $<$ in the definition is quasi-linear.

For finite U, B-relations $R \subset P^+(U) \times P^+(U)$ can also be defined by the modified formula:

$$\text{(RNB')}\ pRq \ \Leftrightarrow\ \forall y(q(y) \to \exists x(p(x) \wedge x < y)),$$

for a quasi-linear relation $<$; again, $<=<_R$. So, (RNB) is equivalent to (RNB'), but this equivalence is not purely logical: it uses the ordering properties of $<$ and the finiteness of U.

Theorem 6 *Let U be finite, and let $R \subset P^+(U) \times P^+(U)$. Then, R satisfies (C1)-(C3) if, and only if, R is defined by (RNB') with $<$ quasi-linear.*

Proof. ($\Leftarrow$). Assume (RNB') with $<$ quasi-linear. We define the relation $R_<$, by (RNB) (strictly, we restrict $R_<$ to nonempty arguments). By logic, $R_< \subset R$. We show $R \subset R_<$. Let pRq. Let $x_1, \ldots, x_n$ be all elements of p such that, for some $y \in q$, there holds $x < y$. Since $<$ is quasi-linear, then there is $j = 1, \ldots, n$ such that $x_j^<$ contains all cones $x_i^<$, for $i = 1, \ldots, n$. Clearly, we obtain:

$$p(x_j) \wedge \forall y(q(y) \to x_j < y),$$

which yields $pR_<q$. Thus, $R = R_<$, hence R satisfies (C1)-(C3), by theorem 1.

($\Rightarrow$). Assume (C1)-(C3). By theorem 1, $R = R_<$, with $<=<_R$ being quasi-linear. Let $R^<$ be defined by (RNB'). As above, we show $R = R^<$. □

The remaining three kinds of relation can also be defined by the following formulas:

(M$B^{\smile}$) $\forall y(p(y) \to \exists x(q(x) \wedge x < y))$,

(MB^{c}) $\exists y(q(y) \wedge \forall x(p(x) \to x \not< y))$,

(M$B^{\smile c}$) $\exists y(p(y) \wedge \forall x(q(x) \to x \not< y))$,

with $<$ quasi-linear.

Observe that theorem 6 cannot be extended to infinite universes. For relations defined by (RNB') with $<$ quasi-linear are completely anti-additive in the second argument, that means, they fulfill the equivalence in (IC2), for arbitrary p, whereas relations defined by (RNB) require a restriction of (IC2) (see the discussion in section 4 and fact 5). Accordingly, for infinite universes, (RNB') and (RNB) give rise to two different classes of relations. From the linguistic point of view, (RNB) seems to be a better approximation of the meaning of *before*, since (RNB') is non-veridical in the first argument (take $q = \emptyset$). From the mathematical point of view, however, (RNB') generates an interesting class of relations connected with quasi-linear orderings, and the reader is invited to find a characterization of these relations in terms of additiveness.

At the end, we briefly discuss other kinds of temporal relations. The connective *after*, considered in section 1, can be represented by the relation:

(RA) $pRq \;\Leftrightarrow\; \exists x(p(x) \wedge \exists y(q(y) \wedge y < x))$.

Clearly, relations defined by (RA) are (completely) additive in both arguments, that means, they satisfy (C1) and (C1') ((IC1) and (IC1')). They also fulfill *double veridicality*:

(DC0) $pRq \;\Rightarrow\; p \neq \emptyset \wedge q \neq \emptyset$.

Using arguments similar to those given above, one shows that, for finite universes U, conditions (C1), (C1') and (DC0) determine precisely the relations defined by (RA) with $<=<_R$, and an analogous characterization can be obtained for infinite universes. However, no information about the relation $<$ can be inferred. If one additionally assumes $\neg\{x\}R\{x\}$, then she gets the irreflexivity of $<$. Thus, the additiveness properties of *after* provide no interesting information about the structure of time.

Kamp [11] proposes the following definitions for *since* and *until* (precisely, we simplify Kamp's formulas in the same way, as (RNB) has simplified (NB)):

(RS) $pRq \;\Leftrightarrow\; \exists x(q(x) \wedge \forall y(x < y \to p(y)))$, (RU) $pRq \;\Leftrightarrow\; \exists x(q(x) \wedge \forall y(y < x \rightarrow p(y)))$.

As observed in Sanchez Valenzia et al. [15], both relations are additive in the second argument, by logic, and, if $<$ is a linear ordering, then they are also multiplicative in the first argument:

(C5) $(p \wedge p')Rq \;\Leftrightarrow\; (pRq) \wedge (p'Rq)$.

Again, we may replace linearity by quasi-linearity:

Fact 6 *Let R be defined by (RS) (resp. (RU)). Then, R satisfies (C5) if, and only if, $<$ satisfies (QC).*

Proof. We only consider the case of (RS), since the other case is dual (use fact 2). Assume R satisfy (C5). Let $x, y \in U$. Set $q = \{x, y\}$, $p = x^{<}$, $p' = y^{<}$. Clearly, pRq and $p'Rq$ hold true. By (C5), we get $(p \wedge p')Rq$. Thus, one of the following conditions holds:

$$\forall z(x < z \rightarrow p(z) \wedge p'(z)),\; \forall z(y < z \rightarrow p(z) \wedge p'(z)),$$

and they are equivalent to:

$$x^{<} \subset x^{<} \cap y^{<},\; y^{<} \subset x^{<} \cap y^{<},$$

which yields $x^{<} \subset y^{<}$ or $y^{<} \subset x^{<}$. Now, assume that $<$ satisfy (QC). We prove the right-to-left direction of (C5) (the other direction is logical). Let pRq and $p'Rq$. Then, there are $x_1, x_2 \in q$ such that:

$$\forall y(x_1 < y \rightarrow p(y)),\; \forall y(x_2 < y \rightarrow p'(y)).$$

If $x_1^{<} \subset x_2^{<}$, then there holds:

$$q(x_1) \wedge \forall y(x_1 < y \rightarrow p(y) \wedge p'(y)),$$

and, if $x_2^{<} \subset x_1^{<}$, then in the latter formula we take x_2 instead of x_1. Consequently, $(p \wedge p')Rq$ holds. □

A detailed analysis of relations connected with *since* and *until* is left to further research. Notice that the equivalence:

$$x < y \;\Leftrightarrow\; \{x\}R\{y\},$$

which is plausible for *before* and *after*, is not plausible for *since* and *until*, hence one cannot represent $<$ by $<_R$. Accordingly, our methods of proofs for characterization results admit no direct counterparts for the latter connectives. It seems that relations defined by (RS), (RU) can only be characterized by conditions expressed in terms of $(U, <)$ rather than the plane U. Yet, notice the interesting equivalence of (RU) and (MB^c) (with $\neg p$ substituted for p). If R' is defined by (MB^c), and R is defined by (RU), then:

$$pRq \;\Leftrightarrow\; (\neg p)R'q.$$

Thus, at least for nonempty events on finite universes (see theorem 6), we obtain the equivalence:

$$p \textit{ until } q \Leftrightarrow \textit{it is not the case that } \neg p \textit{ before } q,$$

for quasi-linear orderings $<$. Further, negating the first argument in B^c relations changes (C2') into (C5), which suggests a way for obtaining characterization results for (RU) from those for (RNB) by more sophisticated considerations.

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