Interpolation and FEP for Logics of Residuated Algebras

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Abstract

A residuated algebra (RA) is a generalization of a residuated groupoid; instead of one basic binary operation $\cdot$ with residual operations $\backslash, /$, it admits finitely many basic operations, and each $n-$ary basic operation is associated with n residual operations. A logical system for RAs was studied in e.g. [6, 8, 16, 15] under the name: Generalized Lambek Calculus **GL**. In this paper we study **GL** and its extensions in the form of sequent systems. We prove an interpolation property which allows to replace a substructure of the antecedent structure by a single formula in a provable sequent. Together with model constructions, based on nuclei [13], interpolation leads to proofs of Finite Embeddability Property for different classes of RAs, as e.g. all RAs, distributive lattice-ordered RAs, boolean RAs, Heyting RAs and double RAs.

1 Introduction

A residuated groupoid is an (ordered) algebra $(M, \cdot, \backslash, /, \leq)$ such that $(M, \leq)$ is a poset, and $\cdot, \backslash, /$ are binary operations on M, satisfying the residuation law:

$$a \cdot b \leq c \text{ iff } b \leq a\backslash c \text{ iff } a \leq c/b\,, \tag{1}$$

for all $a, b, c \in M$. One refers to $\cdot$ as product, and to $\backslash$ (resp. $/$) as the right (resp. left) residual operation for product. (1) implies the monotonicity of product in both arguments. Actually, product is distributive over (existing) infinite joins [13].

Instead of single product $\cdot$ one may admit a finite number of finitary operations o. With each $n-$ary operation o there are associated n *residual operations* o/i, for $i = 1, \ldots, n$, which satisfy the general residuation law:

$$o(a_1, \ldots, a_n) \leq b \text{ iff } a_i \leq (o/i)(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)\,, \tag{2}$$

for all $a_1, \ldots, a_n, b \in M$. This leads to the notion of *a residuated algebra* (RA). A nullary operation o gives rise to no residual operations. For $n = 1$, (2) admits the form:

$$o(a) \leq b \text{ iff } a \leq (o/1)(b)\,, \tag{3}$$

which is a kind of Galois correspondence; $o/1$ is the right adjoint of o. The pair $o, o/1$ can be treated as substructural unary modalities [21]. As a consequence of (2), one easily proves:

$$o(a_1, \ldots, a_{i-1}, \bigvee_{j \in J} b_j, \ldots, a_n) = \bigvee_{j \in J} o(a_1, \ldots, a_{i-1}, b_j, \ldots, a_n)\,, \tag{4}$$

$$(o/i)(a_1, \ldots, a_{i-1}, \bigwedge_{j \in J} b_j, \ldots, a_n) = \bigwedge_{j \in J} (o/i)(a_1, \ldots, a_{i-1}, b_j, \ldots, a_n)\,, \tag{5}$$

and for $a_k = \bigvee\{b_j : j \in J\}$, $k \neq i$,

$$(o/i)(a_1, \ldots, a_k, \ldots, a_n) = \bigwedge_{j \in J} (o/i)(a_1, \ldots, a_{k-1}, b_j, \ldots, a_i, \ldots, a_n)\,, \tag{6}$$

if the infinite joins and meets on the left-hand side exist. Consequently, operations o are monotone in each argument, and o/i is monotone in $i-$th argument and antitone in every $j-$th argument, for $j \neq i$.

An RA with lattice operations $\vee, \wedge$ is called *a lattice-ordered RA*. It is said to be *distributive*, if its lattice reduct is distributive. *A boolean RA* is an RA with operations $\vee, \wedge, \neg$ such that the $(\vee, \wedge, \neg)-$reduct is a boolean algebra; the lower bound is denoted by $\perp$ and the upper bound by $\top$. In a similar way we define *a Heyting RA* with $\vee, \wedge, \rightarrow, \perp, \top$ (here $\rightarrow$ is pseudo-complement, i.e. the residual operation for $\wedge$). For each case, the lattice (boolean) ordering is the ordering of the RA.

Lattice-ordered residuated monoids, also called residuated lattices, are a basic class of algebras, modeling *substructural logics*, i.e. logics devoid of some structural rules (Weakening, Contraction, Exchange) [20, 21, 13]. Famous substructural logics are Linear Logics and its fragments. In the context of Linear Logics, one refers to product and its residuals (and their duals) as *multiplicatives* and to $\vee, \wedge$ as *additives* . The Lambek calculus is the multiplicative fragment of Intuitionistic Noncommutative Linear Logic (precisely, we mean system **L1**, the Lambek calculus with 1 [9]).

Nonassociative Lambek Calculus **NL**, introduced by Lambek [18], is the complete logic of residuated groupoids. Its algebraic form admits sequents $\alpha \Rightarrow \beta$, where α, β are formulas formed out of variables and operation symbols $\cdot, \backslash, /$. The axioms are all sequents $\alpha \Rightarrow \alpha$, and the inference rules are: (1) from $\alpha \cdot \beta \Rightarrow \gamma$ infer $\beta \Rightarrow \alpha\backslash\gamma$, and conversely, (2) from $\alpha \cdot \beta \Rightarrow \gamma$ infer $\alpha \Rightarrow \gamma/\beta$, and conversely, (3) from $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ infer $\alpha \Rightarrow \gamma$. Clearly (3) is a variant of the cut rule; it cannot be eliminated in this system. Lambek [18] also formulated a sequential system of **NL** which admits cut elimination (see section 3); it yields the decidability of **NL**. The sequential system admits sequents $X \Rightarrow \alpha$ such that α is a formula, and X is a bracketed string (a tree) of formulas.

In [8], it is proved that the consequence relation of **NL** is decidable in polynomial time. The proof employs an interpolation property of the sequential form of **NL**: if $X[Y] \Rightarrow \alpha$ is derivable in **NL** from a set of assumptions Φ, then there exists a formula δ such that $X[\delta] \Rightarrow \alpha$ and $Y \Rightarrow \delta$ are derivable in **NL** from Φ; furthermore, δ is a subformula of some formula appearing in $X[Y] \Rightarrow \alpha$ or Φ. An analogous (slightly weaker) property of the pure **NL** has been first proved by Jäger [14] by induction on cut-free proofs in **NL**. The proof in [8] is different. One works with the cut rule (it is necessary for systems with assumptions); first, a subformula property is proved by model-theoretic tools, then, interpolation and the polynomial time decision method are obtained by a direct construction.

In [12], this interpolation property is applied to prove Strong Finite Model Property (SFMP) for **NL** and **NL** with additive conjunction $\wedge$. It yields Finite Embeddability Property (FEP) for the class of residuated groupoids (solving a problem in [4]). [12] also contains analogous results for **NL** with lattice operations $\wedge, \vee$, satisfying the distributive law; these are common results of M. Farulewski and the present author. In [10], these methods are applied to prove that categorial grammars based on **NL** with distributive lattice operations generate context-free languages. At the end of the latter paper, there are announced (without proof) analogous results (due to the present author) for **NL** with operations of boolean algebra or Heyting algebra and a generalized version of **NL** with several product operations.

The present paper elaborates the latter subject in detail. We believe that the general framework of RAs is interesting for both logic and its applications. Algebras with several basic multiplicative operations appear in different areas of logic, e.g. multi-modal logics and abstract algebraic logics. The gaggle theory of Dunn [11] provides a Kripke-style relational semantics for such logics. Different variants of the Lambek calculus are widely applied as type change logics for categorial grammars (see e.g. [2, 7] and the chapters of M. Moortgat and the present author in [3]). A standard representation of linguistic expressions is a labeled tree, which can be represented as a term of a formal language with operations corresponding to vertex labels. These applications are not exploited here, since this paper focuses on some purely logical problems.

We provide a complete proof of FEP for the class of distributive lattice-ordered RAs (of an arbitrary signature) and a number of related results. In section 2 we present basic constructions of lattice-ordered RAs: the powerset construction and the nucleus construction; they modify and extend standard constructions in the literature (see [19, 1, 13, 12]), but our treatment of the nucleus construction is slightly different. In section 3 we formulate a sequential system of Generalized Lambek Calculus **GL**, a complete logic of RAs, and its variants with lattice operations **FGL** and the distributive law **DFGL**; we also prove an interpolation lemma for each of these systems. The main result is proved in section 4; Theorem 1 states that **DFGL** possesses SFMP (it yields FEP for the corresponding class of algebras). In remarks, we show that FEP also holds for the classes of all RAs and all RAs with $\wedge$.

From the point of view of this paper, the move from residuated groupoids to RAs is not merely a generalization for itself. Besides the applications, mentioned above, algebras with several basic operations are essential for our treatment of boolean RAs and Heyting RAs. FEP for these classes is proved in section 5.

Analogous results are obtained for unital RAs and integral RAs. Since proofs are similar to those for **DFGL**, we only discuss the relevant differences and additions. (Theorem 1 is the only designated theorem of this paper, but it splits in numerous variants.) At the end, we briefly outline analogous results for double RAs, i.e. RAs in which some operations satisfy (2) with respect to $\geq$, not $\leq$ (here the RA-framework is quite natural, again). As a special case, we mention algebras of modal logics.

In a forthcoming paper we plan to elaborate on the case of modal algebras and to handle proof-theoretic decision methods and complexity problems.

Recall that SFMP for a class of algebras $\mathcal{K}$ amounts to Finite Model Property (FMP) for the Horn theory of $\mathcal{K}$. If $\mathcal{K}$ is closed under finite products (including the trivial product, which yields the trivial algebra), then SFMP is equivalent to FEP: every finite, partial subalgebra of an algebra from $\mathcal{K}$ can be embedded in some finite algebra from $\mathcal{K}$ (FEP for $\mathcal{K}$ is equivalent to FMP for the universal theory of $\mathcal{K}$). If a formal system **S** is strongly complete with respect to $\mathcal{K}$, then it yields, actually, an axiomatization of the Horn theory of $\mathcal{K}$; hence SFMP for **S** with respect to $\mathcal{K}$ yields SFMP for $\mathcal{K}$.

2 Residuated algebras

RAs were defined in section 1. By $\mathcal{M}$ and M we denote an algebra and its universe, respectively. σ denotes the signature of an algebra $\mathcal{M}$ without residual operations (*a basic algebra*) and σ_r the extended signature with residual operations. For example, if $\mathcal{M} = (M, o, o')$ is an algebra of signature $\sigma = (1, 2)$, then $\sigma_r = (1, 1, 2, 2, 2)$, which corresponds to unary operations $o, o/1$ and binary operations $o', o'/1, o'/2$. The signature may also admit nullary operations (interpreted as designated elements of M); no residual operations are defined for them. We need two general constructions of RAs.

The first one leads from an algebra $\mathcal{M}$ of signature σ to *the powerset algebra* $P(\mathcal{M})$ of signature σ_r. The universe of $P(\mathcal{M})$ is the powerset $P(M)$. Subsets of M are denoted by U, V, W (possibly with indices; we omit this remark in what follows). Elements of M are denoted by a, b, c. Each $n-$ary operation o of $\mathcal{M}$ determines an operation O in $P(\mathcal{M})$:

$$O(U_1, \ldots, U_n) = \{o(a_1, \ldots, a_n) : a_1 \in U_1, \ldots, a_n \in U_n\}\,. \tag{7}$$

If $n = 0$, then $O = \{o\}$. For $n \geq 1$, the residual operations for O are defined as follows:

$$(O/i)(U_1, \ldots, U_n) = \{a \in M : O(U_1, \ldots, \{a\}, \ldots, U_n) \subseteq U_i\}\,, \tag{8}$$

for $i = 1, \ldots, n$. It is clear from the context that $\{a\}$ is the $i-$th argument of O/i, and we omit the adjacent arguments U_{i-1}, U_{i+1}. Clearly, for $n = 1$, $O(U) = o[U]$ and $(O/1)(U) = o^{-1}[U]$ (the image and the co-image, respectively, of U under o). The order is $\subseteq$. We leave to the reader the routine proof of (2). So, $P(\mathcal{M})$ is an RA. $P(M)$ is a complete lattice of sets, whence a distributive lattice. In

[16], using a labeled deductive system in the style of [5], it is shown that every RA (without nullary operations) can be embedded into some powerset algebra, but this embedding need not preserve lattice operations.

The second construction leads from an algebra $\mathcal{M}$ of signature σ via $P(\mathcal{M})$ to an algebra of closed subsets of M (an RA of signature σ_r). An operator $C : P(M) \mapsto P(M)$ is called *a closure operator* or *a nucleus* on $\mathcal{M}$, if it satisfies the following conditions: (C1) $U \subseteq C(U)$, (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$, (C3) $C(C(U)) \subseteq C(U)$, (C4) $O(C(U_1), \ldots, C(U_n)) \subseteq C(O(U_1, \ldots, U_n))$, for all $U, V, U_1, \ldots, U_n \subseteq M$ and all non-nullary operations O of $P(\mathcal{M})$. A similar notion of a closure operator on a monoid is used in constructions of phase-space models for intuitionistic fragments of Linear Logic [19, 1, 13].

Let C be a closure operator on $\mathcal{M}$. A set $U \subseteq M$ is said to be *closed*, if $C(U) = U$. $C[\mathcal{M}]$ denotes the family of $C-$closed subsets of M. By (C1)-(C3), $C[\mathcal{M}]$ is closed under arbitrary meets, whence it is a complete lattice with order $\subseteq$ (in general, it is not distributive). As a consequence of (2), one gets:

$$o(a_1, \ldots, a_{i-1}, (o/i)(a_1, \ldots, a_n), \ldots, a_n) \leq a_i \,, \tag{9}$$

for any elements and operations of the given RA. We use (9) to show that $C[\mathcal{M}]$ is closed under residual operations O/i.

Lemma 1. *For any $U_1, \ldots, U_n \subseteq M$ and any $i = 1, \ldots, n$, if U_i is closed, then $(O/i)(U_1, \ldots, U_n)$ is closed.*

Proof. Operations O are monotone in all arguments. One calculates:

$$O(U_1, \ldots, C((O/i)(U_1, \ldots, U_n)), \ldots, U_n) \subseteq$$

$$O(C(U_1), \ldots, C((O/i)(U_1, \ldots, U_n)), \ldots, C(U_n)) \subseteq$$

$$C(O(U_1, \ldots, (O/i)(U_1, \ldots, U_n), \ldots, U_n)) \subseteq C(U_i) = U_i \,,$$

whence $C((O/i)(U_1, \ldots, U_n)) \subseteq (O/i)(U_1, \ldots, U_n)$. □

$C[\mathcal{M}]$ is an algebra of signature σ_r with operations: $O_C(U_1, \ldots, U_n) = C(O(U_1, \ldots, U_n))$, for $n \geq 0$, and O_C/i equal to O/i restricted to $C[\mathcal{M}]$, for $n \geq 1$, $i = 1, \ldots, n$. It is easy to see that $C[\mathcal{M}]$ is an RA. We also define lattice operations and bounds: $U \wedge V = U \cap V$, $U \vee_C V = C(U \cup V)$, $\bot = C(\emptyset)$, $\top = M$. $C[\mathcal{M}]$ with these operations and constants is a complete lattice (not necessarily distributive).

Lemma 2. *Let $\mathcal{M}$ be an algebra of signature σ, and let C be an operator on $P(M)$, satisfying (C1)-(C3). Then, C satisfies (C4) if and only if the condition of Lemma 1 is true for $C[\mathcal{M}]$.*

Proof. The 'only if' part is just Lemma 1. For the 'if' part, by induction on $i = 0, \ldots, n$, we prove:

$$O(C(U_1), \ldots, C(U_i), U_{i+1}, \ldots, U_n) \subseteq C(O(U_1, \ldots, U_n)) \,, \tag{10}$$

for any $U_1, \ldots, U_n \subseteq M$, assuming that the condition of Lemma 1 holds. For $i = 0$, (10) follows from (C1). Assume that (10) holds for $i < n$. By (2) for $P(\mathcal{M})$, we get:

$$U_{i+1} \subseteq (O/i+1)(C(U_1), \ldots, C(U_i), C(O(U_1, \ldots, U_n)), U_{i+2}, \ldots, U_n)\,.$$

The right-hand set is closed, whence the left-hand set can be replaced with $C(U_{i+1})$, by (C2), (C3). We apply (2) for $P(\mathcal{M})$ and obtain (10) for $i+1$. $\square$

We will use a closure operator determined by a family $\mathcal{B}$ of subsets of M. One defines:

$$C_{\mathcal{B}}(U) = \bigcap\{V \in \mathcal{B} : U \subseteq V\}\,, \tag{11}$$

for $U \subseteq M$.

$C_{\mathcal{B}}$ satisfies (C1)-(C3), for any family $\mathcal{B}$. Again, the family $C_{\mathcal{B}}[\mathcal{M}]$ is closed under arbitrary meets.

Lemma 3. *$C_{\mathcal{B}}$ satisfies (C4) if and only if, for all non-nullary operations o in $\mathcal{M}$, all sets $V \in \mathcal{B}$ and all $a_1, \ldots, a_n \in M$, the set $(O/i)(\{a_1\}, \ldots, V, \ldots, \{a_n\})$ is closed, for every $i = 1, \ldots, n$.*

Proof. The 'only if' part follows from the fact that all sets in $\mathcal{B}$ are closed, applying Lemma 2. The 'if' part is a consequence of (5) and (6), applied to $P(\mathcal{M})$ and Lemma 2. $\square$

3 Generalized Lambek Calculus

We present a formal system which proves order formulas $\alpha \leq \beta$ valid in RAs. First, we recall **NL** as a sequent system [18].

We admit a denumerable set of variables $p, q, r, \ldots$. Formulas are built from variables by means of $\cdot, \backslash, /$. Formula structures (shortly: structures) are built from formulas according to the rule: if X, Y are structures then (X, Y) is a structure. We denote arbitrary formulas by $\alpha, \beta, \gamma, \ldots$ and structures by X, Y, Z. *Contexts* are structures which contain a unique occurrence of a new atomic substructure $\circ$; they are denoted $X[\circ], Y[\circ], Z[\circ]$ etc. If $X[\circ]$ is a context and Y is a structure, then $X[Y]$ denotes the substitution of Y for $\circ$ in $X[\circ]$.

Sequents are of the form $X \Rightarrow \alpha$; in models $\Rightarrow$ is interpreted as $\leq$. **NL** (in the sequential form, due to Lambek [18]) assumes the following axioms and inference rules:

$$\text{(Id) } \alpha \Rightarrow \alpha\,,$$

$$(\cdot\text{L})\ \frac{X[(\alpha, \beta)] \Rightarrow \gamma}{X[\alpha \cdot \beta] \Rightarrow \gamma},\ (\cdot\text{R})\ \frac{X \Rightarrow \alpha;\ Y \Rightarrow \beta}{(X, Y) \Rightarrow \alpha \cdot \beta},$$

$$(\backslash\text{L})\ \frac{X[\beta] \Rightarrow \gamma;\ Y \Rightarrow \alpha}{X[(Y, \alpha\backslash\beta)] \Rightarrow \gamma},\ (\backslash\text{R})\ \frac{(\alpha, X) \Rightarrow \beta}{X \Rightarrow \alpha\backslash\beta},$$

$$(/\text{L})\ \frac{X[\beta] \Rightarrow \gamma;\ Y \Rightarrow \alpha}{X[(\beta/\alpha, Y)] \Rightarrow \gamma},\ (/\text{R})\ \frac{(X, \alpha) \Rightarrow \beta}{X \Rightarrow \beta/\alpha},$$

$$\text{(CUT)}\ \frac{X[\alpha] \Rightarrow \beta;\ Y \Rightarrow \alpha}{X[Y] \Rightarrow \beta}.$$

In Full Nonassociative Lambek Calculus **FNL**, $\cdot, \backslash, /$ are enriched with $\wedge, \vee$, and one admits the following rules:

$$(\wedge\text{L})\ \frac{X[\alpha_i] \Rightarrow \beta}{X[\alpha_1 \wedge \alpha_2] \Rightarrow \beta},\ (\wedge\text{R})\ \frac{X \Rightarrow \alpha;\ X \Rightarrow \beta}{X \Rightarrow \alpha \wedge \beta},$$

$$(\vee\text{L})\ \frac{X[\alpha] \Rightarrow \gamma;\ X[\beta] \Rightarrow \gamma}{X[\alpha \vee \beta] \Rightarrow \gamma},\ (\vee\text{R})\ \frac{X \Rightarrow \alpha_i}{X \Rightarrow \alpha_1 \vee \alpha_2},$$

In ($\wedge$L) and ($\vee$R), the subscript i equals 1 or 2. The latter rules and ($\cdot$L), ($\backslash$R), (/R) have one premise; the remaining rules have two premises, separated by semicolon.

Generalized Lambek Calculus **GL** replaces $\cdot, \backslash, /$ with operation symbols $o, o/i$ from a signature σ_r. (So, it is more correct to write $\mathbf{GL}_\sigma$, but the subscript will be omitted.) The notion of a formula is defined in a natural way. The algebraic form of **GL** admits sequents of the form $\alpha \Rightarrow \beta$. The axioms are all sequents $\alpha \Rightarrow \alpha$. The inference rules are: (1) from $o(\alpha_1, \ldots, \alpha_n) \Rightarrow \alpha$ infer $\alpha_i \Rightarrow (o/i)(-)$, and conversely, for $i = 1, \ldots, n$, where $(-)$ stands for the sequence $(\alpha_1, \ldots, \alpha_n)$ in which α_i has been replaced by α, (2) from $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \gamma$ infer $\alpha \Rightarrow \gamma$.

We need a sequential system for **GL**. Formula structures employ a structure constructor $(-, \ldots, -)_o$, for any operation symbol o. The recursive definition is as follows: (i) all formulas are structures, (ii) if o is an $n-$ary operation symbol ($n \geq 0$) and $X_1, \ldots, X_n$ are structures, then $(X_1, \ldots, X_n)_o$ is a structure. Notice that we admit the 'empty' structure $()_o$, for any nullary operation symbol o. The axioms of **GL** are (Id). The rules are natural counterparts of the rules of **NL**:

$$(o\text{L})\ \frac{X[(\alpha_1, \ldots, \alpha_n)_o] \Rightarrow \gamma}{X[o(\alpha_1, \ldots, \alpha_n)] \Rightarrow \gamma},$$

$$(o\text{R})\ \frac{X_1 \Rightarrow \alpha_1; \ldots; X_n \Rightarrow \alpha_n}{(X_1, \ldots, X_n)_o \Rightarrow o(\alpha_1, \ldots, \alpha_n)},$$

$$(o/i\text{L})\ \frac{X[\alpha_i] \Rightarrow \gamma; Y_1 \Rightarrow \alpha_1; \ldots; Y_n \Rightarrow \alpha_n}{X[(Y_1, \ldots, (o/i)(\alpha_1, \ldots, \alpha_n), \ldots, Y_n)_o] \Rightarrow \gamma},$$

$$(o/i\text{R})\ \frac{(\alpha_1, \ldots, X, \ldots, \alpha_n)_o \Rightarrow \alpha_i}{X \Rightarrow (o/i)(\alpha_1, \ldots, \alpha_n)}.$$

The system contains (CUT). Rules (oL), (oR) are also admitted for nullary operation symbols o; then, (oR) is actually an axiom $()_o \Rightarrow o$. Rules ((o/iL), (o/iR) are admitted for non-nullary operation symbols only. In (o/iL), the sequent $Y_i \Rightarrow \alpha_i$ does not appear among premises. In the premise of (o/iR), X is the $i-$th constituent of $(-, \ldots, -)_o$.

Full Generalized Lambek Calculus **FGL** admits $\wedge, \vee$, and the rules for them are as above. Clearly **NL** (resp. **FNL**) equals **GL** (resp. **FGL**) with $\sigma = (2)$.

Both **GL** and **FGL** admit cut elimination. For **NL**, the proof was already in Lambek [18]; the proof for **GL** is similar and can be adapted to **FGL** (also see [20, 13]).

It is easy to prove that **GL** (resp. **FGL**) is strongly complete with respect to (resp. lattice-ordered) RAs. We recall some basic notions. *An assignment* in an RA $\mathcal{M}$ is a homomorphism from the formula algebra into $\mathcal{M}$. An assignment μ can be defined on formula structures: $\mu(X) = \mu(F(X))$, where $F(X)$ arises from X, after one has successively replaced every substructure $(X_1, \ldots, X_n)_o$ by the formula $o(F(X_1), \ldots, F(X_n))$; in particular, $()_o$ is replaced by o, if o is nullary. A sequent $X \Rightarrow \alpha$ is *true* in the model $(\mathcal{M}, \mu)$, if $\mu(X) \leq \mu(\alpha)$. Let Φ be a set of sequents. Sequents in Φ will be treated as *assumptions*. Since $(X_1, \ldots, X_n)_o \Rightarrow \alpha$ is deductively equivalent to $o(F(X_1), \ldots, F(X_n)) \Rightarrow \alpha$, then we assume that every sequent from Φ is of the form $\alpha \Rightarrow \beta$ (such sequents are said to be *simple*). $\Phi \vdash X \Rightarrow \alpha$ means: $X \Rightarrow \alpha$ is provable in the calculus (indicated by the context) from the set of assumptions Φ. The strong completeness means the following: $\Phi \vdash X \Rightarrow \alpha$ in **GL** if and only if, for any model $(\mathcal{M}, \mu)$, if all sequents from Φ are true, then $X \Rightarrow \alpha$ is true. Here $\mathcal{M}$ ranges over RAs. For **FGL**, an analogous equivalence is true, with $\mathcal{M}$ ranging over lattice-ordered RAs.

We are concerned with **FGL** enriched with the distributive law for lattice operations. It is added as a new axiom:

$$\text{(D) } \alpha \wedge (\beta \vee \gamma) \Rightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$

for all formulas α, β, γ. Notice that the converse sequent is provable in **FGL** (it is true in every lattice). The resulting system is denoted **DFGL**. This system is strongly complete with respect to distributive lattice-ordered RAs.

(CUT) can be eliminated in **FGL** but not in **DFGL**. An equivalent cut-free system can also be designed, following J.M. Dunn and G. Mints (see [21]), with a special structure constructor corresponding to $\wedge$. This way, however, leads to some technical complications. We prefer to admit (CUT). Actually, one of the most characteristic features of our approach is that we proceed with (CUT) without lacking effectiveness.

Cut elimination is replaced by *interpolation* of a particular kind. In a provable sequent $X[Y] \Rightarrow \alpha$, the subtree Y can be replaced by its *interpolant* δ such that $X[\delta] \Rightarrow \alpha$ and $Y \Rightarrow \delta$ are provable. Restricted sets of sequents lead to finite sets of interpolants. This is essential in our proofs of FEP for distributive lattice-ordered RAs and other classes. It also plays a crucial role in proof-theoretic decision procedures for **DFGL** and its variants with assumptions (we briefly comment on them at the end of the next section).

We prove our first interpolation lemma. Φ denotes a set of assumptions. Let T denote a set of formulas. By a $T-$sequent we mean a sequent such that all formulas occurring in it belong to T. We write $\Phi \vdash_T X \Rightarrow \alpha$ if $X \Rightarrow \alpha$ has a deduction from Φ (in the given calculus) which consists of $T-$sequents only (called a $T-$deduction).

Lemma 4. *Let T be closed under $\wedge, \vee$. Let $\Phi \vdash_T X[Y] \Rightarrow \gamma$ in* **DFGL**. *Then, there exists $\delta \in T$ such that $\Phi \vdash_T X[\delta] \Rightarrow \gamma$ and $\Phi \vdash_T Y \Rightarrow \delta$ in* **DFGL**.

Proof. Let Y, Z be substructures of a structure X (in this proof, by a substructure we always mean an occurrence of a substructure). We say that Y contains Z, if Z is a substructure of Y. Clearly any two substructures of X are either disjoint (they have no common atomic substructure), or one of them contains the other.

The proof proceeds by induction on $T-$deductions of $X[Y] \Rightarrow \gamma$ from Φ. The case of axioms and assumptions is easy; they are simple sequents $\alpha \Rightarrow \gamma$, so $Y = \alpha$ and $\delta = \alpha$.

Let $X[Y] \Rightarrow \gamma$ be the conclusion of a rule. (CUT) is easy. If Y comes from one premise of (CUT), i.e. Y is contained in the antecedent of the premise, then we take an interpolant from this premise. Otherwise $Y = U[Z]$, where the premises are $X[U[\alpha]] \Rightarrow \gamma$, $Z \Rightarrow \alpha$. Then, an interpolant δ of $U[\alpha]$ in the first premise is also an interpolant of Y in the conclusion, by (CUT).

Let us consider other rules. First, we assume that Y does not contain the formula, introduced by the rule (the active formula). If Y comes from exactly one premise of the rule, i.e. Y is contained in the antecedent of one premise, but not in the antecedent of any other premise (if they exist), then one takes an interpolant of Y in this premise. Let us consider ($\wedge$R). The premises are $X[Y] \Rightarrow \alpha$, $X[Y] \Rightarrow \beta$, and the conclusion is $X[Y] \Rightarrow \alpha \wedge \beta$. By the induction hypothesis, there are interpolants δ of Y in the first premise and δ' of Y in the second one. We have $\Phi \vdash_T X[\delta] \Rightarrow \alpha$, $\Phi \vdash_T X[\delta'] \Rightarrow \beta$, $\Phi \vdash_T Y \Rightarrow \delta$, $\Phi \vdash_T Y \Rightarrow \delta'$. Then, $\delta \wedge \delta'$ is an interpolant of Y in the conclusion, by ($\wedge$L), ($\wedge$R). Let us consider ($\vee$L). The premises are $X[\alpha][Y] \Rightarrow \gamma$, $X[\beta][Y] \Rightarrow \gamma$, and the conclusion is $X[\alpha \vee \beta][Y] \Rightarrow \gamma$, where Y does not contain $\alpha \vee \beta$. As above, there are interpolants δ, δ' of Y in the premises. Again $\delta \wedge \delta'$ is an interpolant of Y in the conclusion, by ($\wedge$L), ($\vee$L) and ($\wedge$R). For (oR) with premises $X_i \Rightarrow \alpha_i$ and the conclusion $(X_1, \ldots, X_n)_o \Rightarrow o(\alpha_1, \ldots, \alpha_n)$, if $Y = (X_1, \ldots, X_n)_o$, then we take $\delta = o(\alpha_1, \ldots, \alpha_n)$.

Second, we assume that Y contains the active formula (so, the rule must be an L-rule). If Y is a single formula, then we take $\delta = Y$. Assume that Y is not a formula. For (oL), ($\wedge$L), we take an interpolant of Y' in the premise, where Y' is the natural source of Y; for (oL), it means that the premise is $X[U[(\alpha_1, \ldots, \alpha_n)_o]] \Rightarrow \gamma$, the conclusion is $X[U[o(\alpha_1, \ldots, \alpha_n)]] \Rightarrow \gamma$, $Y = U[o(\alpha_1, \ldots, \alpha_n)]$, $Y' = U[(\alpha_1, \ldots, \alpha_n)_o]$, and for ($\wedge$L), it means that the premise if $X[U[\alpha_i]] \Rightarrow \gamma$, the conclusion is $X[U[\alpha_1 \wedge \alpha_2]] \Rightarrow \gamma$, $Y = U[\alpha_1 \wedge \alpha_2]$, $Y' = U[\alpha_i]$. For (o/iL) with premises $X[\alpha_i] \Rightarrow \gamma$, $Y_j \Rightarrow \alpha_j$, for $j \neq i$, and the conclusion

$$X[(Y_1, \ldots, (o/i)(\alpha_1, \ldots, \alpha_n), \ldots, Y_n)_o] \Rightarrow \gamma$$

we consider the source Y' of Y. It means that Y contains the substructure $(Y_1, \ldots, Y_n)_o$, designated in the conclusion, and Y' arises from Y by replacing this substructure with α_i. Clearly Y' is contained in the antecedent of the first premise. Hence, an interpolant of Y' in this premise is also an interpolant of Y in the conclusion, by (o/iL). The final case is ($\vee$L) with premises $Z[U[\alpha]] \Rightarrow \gamma$, $Z[U[\beta]] \Rightarrow \gamma$ and the conclusion $Z[U[\alpha \vee \beta]] \Rightarrow \gamma$, where $Y = U[\alpha \vee \beta]$. Let δ be an interpolant of $U[\alpha]$ in the first premise and δ' be an interpolant of $U[\beta]$

in the second premise. Then, $\delta \vee \delta'$ is an interpolant of Y in the conclusion, by ($\vee$L), ($\vee$R). $\square$

REMARKS. (1) The axiom (D) plays no role in the above proof, whence Lemma 4 is true for **FGL**. (2) For the $\vee-$free fragment of **FGL** Lemma 4 remains true except that we assume that T is closed under $\wedge$ only. (3) For **GL**, Lemma 4 is true for an arbitrary set T.

Lemma 4 cannot be proved for associative systems. **FL** can be axiomatized by adding the following associativity rules to **FNL**:

$$\text{(ASS1)}\ \frac{X[((Y_1, Y_2), Y_3)] \Rightarrow \alpha}{X[(Y_1, (Y_2, Y_3))] \Rightarrow \alpha}\ \text{(ASS2)}\ \frac{X[(Y_1, (Y_2, Y_3))] \Rightarrow \alpha}{X[((Y_1, Y_2), Y_3)] \Rightarrow \alpha}.$$

(ASS1) introduces a new substructure (Y_2, Y_3), which does not appear in the premise. An interpolant of (Y_2, Y_3) can be $\delta_2 \cdot \delta_3$, where δ_2 is an interpolant of Y_2 in the premise, and δ_3 is an interpolant of Y_3 in $X[(\delta_2, Y_3)] \rightarrow \alpha$. This idea, however, leads to the requirement that T is closed under $\cdot$, which is undesirable for the purposes of the next section (see Lemma 5 and further). Instead of the above rules, one can add the associative law as the axiom $(\alpha\cdot\beta)\cdot\gamma \Rightarrow \alpha\cdot(\beta\cdot\gamma)$ and the converse one. Then, Lemma 4 remains true (with the same proof), but our proof of Lemma 7 in the next section requires that the set $c(T)$ is closed under all operations which appear in new axioms; if we assume that $c(T)$ is closed under $\cdot$, then Lemma 5 becomes false, whence the finiteness of the algebra in Lemma 7 cannot be proved. Of course, any finite collection of instances of the associative law can be included in Φ without affecting our results.

If Lemma 4 were true for **FL**, then SFMP for **FL** could be proved, by methods of section 4. However, SFMP fails for **FL**, since the consequence relation for **FL** is undecidable. Actually, it is undecidable for the Lambek calculus (even its $/-$fragment) [8], and the consequence relation of **FL** is conservative over that for the Lambek calculus.

The restriction to $T-$deductions can be eliminated (in a sense). In the next section we prove *the subformula property*: if $\Phi \vdash X \Rightarrow \alpha$, then there exists a $T-$deduction of $X \Rightarrow \alpha$ from Φ, for some set T, depending on Φ and $X \Rightarrow \alpha$. As a consequence, we obtain a stronger form of Lemma 4. The proof of the subformula property applies some model-theoretic tools, which yield FEP for **DFGL**.

4 Special models

In this section Φ denotes a fixed finite set of simple sequents. (The finiteness of Φ is not always essential.) T denotes a set of formulas. Two formulas α and β are said to be $T-$*equivalent* in the calculus **S**, if $\Phi \vdash_T \alpha \Rightarrow \beta$ and $\Phi \vdash_T \beta \Rightarrow \alpha$ in **S**. $s(T)$ denotes the closure of T under subformulas, i.e. the smallest set of formulas which contains T and is closed under subformulas. $c(T)$ denotes the closure of T under $\wedge, \vee$. Clearly $c(s(T))$ is the smallest set which contains T and is closed under subformulas and $\wedge, \vee$. The following lemma is closely related to the local finiteness of distributive lattices.

Lemma 5. *If T is a finite set of formulas, then $c(T)$ is finite up to the relation of $c(T)-$equivalence in* **DFGL**.

Proof. Let T be finite. Every formula in $c(T)$ is $c(T)-$equivalent to a finite disjunction of finite conjunctions of formulas from T. Omitting repetitions, we get finitely many formulas of the latter form. □

Remark. For the $\vee-$free fragment of **FGL**, Lemma 5 remains true, if one defines $c(T)$ as the closure of T under $\wedge$. Then, every formula in $c(T)$ is $c(T)-$equivalent to a finite conjunction of formulas from T.

By T^* we denote the set of all formula structures formed out from formulas in T. Similarly, $T^*[\circ]$ denotes the set of all contexts in which any formula belongs to T. T^* is a (free) algebra (of signature σ) with operations defined as follows: $o(X_1, \ldots, X_n) = (X_1, \ldots, X_n)_o$. Notice that the nullary operation o is interpreted as the 'empty' structure $()_o$, not the formula o. We consider the powerset algebra $P(T^*)$. As shown in section 2, $P(T^*)$ is a residuated algebra of signature σ_r.

We define some subsets of T^*. For any $T-$context $X[\circ]$ and any $\alpha \in T$, we define:

$$[X[\circ], \alpha] = \{Y \in T^* : \Phi \vdash_T X[Y] \Rightarrow \alpha \text{ in } \mathbf{DFGL}\},\ [\alpha] = [\circ, \alpha]. \qquad (12)$$

The notation follows [19, 1, 13]. We define $\mathcal{B}(T)$ as the family of all sets $[X[\circ], \alpha]$, for $X[\circ] \in T^*[\circ]$, $\alpha \in T$. The closure operator $C_{\mathcal{B}(T)}$ is defined as in section 2.

Lemma 6. *$C_{\mathcal{B}(T)}$ satisfies (C4).*

Proof. We apply Lemma 3. Let $Y_j \in T^*$, for $j = 1, \ldots, n$, $j \neq i$, and let $[X[\circ], \alpha] \in \mathcal{B}(T)$. We have to show that $(O/i)(\{Y_1\}. \ldots, [X[\circ], \alpha], \ldots, \{Y_n\})$ is closed. Clearly the latter set equals $[X[(Y_1, \ldots, \circ, \ldots, Y_n)_o], \alpha]$, whence it belongs to $\mathcal{B}(T)$. □

Accordingly, the algebra $C_{\mathcal{B}(T)}(P(T^*))$ is a complete lattice-ordered RA; this algebra will be denoted by $\mathcal{M}(T)$.

The following equations are true in $\mathcal{M}(T)$, for any set T, if all formulas appearing in the equation belong to T. We write C for $C_{\mathcal{B}(T)}$.

$$O_C([\alpha_1], \ldots, [\alpha_n]) = [o(\alpha_1, \ldots, \alpha_n)]\,, \qquad (13)$$

$$(O/i)([\alpha_1], \ldots, [\alpha_n]) = [(o/i)(\alpha_1, \ldots, \alpha_n)]\,, \qquad (14)$$

$$[\alpha] \vee_C [\beta] = [\alpha \vee \beta],\ [\alpha] \cap [\beta] = [\alpha \wedge \beta]\,. \qquad (15)$$

We prove (13). To prove ($\subseteq$), observe that, if $X_i \in [\alpha_i]$, $i = 1, \ldots, n$, then $(X_1, \ldots, X_n)_o \in [o(\alpha_1, \ldots, \alpha_n)]$, by ($o$R). Consequently $O([\alpha_1], \ldots, [\alpha_n]) \subseteq$

$[o(\alpha_1,\ldots,\alpha_n)]$, which yields the desired inclusion, by (C2), since the right-hand side is a closed set. To prove ($\supseteq$), assume that $[X[\circ],\alpha] \in \mathcal{B}(T)$ contains $O([\alpha_1],\ldots,[\alpha_n])$. Since $\alpha_i \in [\alpha_i]$, by (Id), we get $(\alpha_1,\ldots,\alpha_n)_o \in [X[\circ],\alpha]$, which means $\Phi \vdash_T X[(\alpha_1,\ldots,\alpha_n)_o] \Rightarrow \alpha$. We get $\Phi \vdash_T X[o(\alpha_1,\ldots,\alpha_n)] \Rightarrow \alpha$, by ($o$L). Using (CUT), one shows $[o(\alpha_1,\ldots,\alpha_n)] \subseteq [X[\circ],\alpha]$, which yields the desired inclusion. Notice that, if o is nullary and $o \in T$, then $O_C = [o]$.

We prove (14). To prove ($\subseteq$), assume $X \in (O/i)([\alpha_1],\ldots,[\alpha_n])$. Since $\alpha_j \in [\alpha_j]$, then $(\alpha_1,\ldots,X,\ldots,\alpha_n)_o \in [\alpha_i]$. By ($o/i$R), $X \in [(o/i)(\alpha_1,\ldots,\alpha_n)]$, as desired. To prove ($\supseteq$), assume $X \in [(o/i)(\alpha_1,\ldots,\alpha_n)]$. The following sequent:

$$(\alpha_1,\ldots,(o/i)(\alpha_1,\ldots,\alpha_n),\ldots,\alpha_n)_o \Rightarrow \alpha_i$$

is provable, by (Id), (o/iL) (this is a $T-$deduction). Hence, by (CUT), for any $Y_j \in [\alpha_j]$, $j \neq i$, the sequent $(Y_1,\ldots,X,\ldots,Y_n)_o \Rightarrow \alpha_i$ is $T-$deducible from Φ. It yields $X \in (O/i)([\alpha_1],\ldots,[\alpha_n])$, as desired.

The proof of (15) is left to the reader (it does not essentially differ from analogous arguments in [1, 13]).

Lemma 7. *Let T be a nonempty finite set of formulas. Then, $\mathcal{M}(c(T))$ is a finite distributive lattice-ordered RA.*

Proof. We denote $T' = c(T)$. By Lemma 5, there exists a finite set $R \subseteq T'$ such that every formula from T' is $T'-$equivalent to some formula from R. A set $U \subseteq (T')^*$ is said to be nontrivial, if $U \neq \emptyset$ and $U \neq (T')^*$. We show that, for any nontrivial closed set U there exists $\alpha \in R$ such that $U = [\alpha]$.

Let U be nontrivial and closed and $X \in U$. Let $[Z[\circ],\beta] \in \mathcal{B}(T')$ contain U. Then $\Phi \vdash_{T'} Z[X] \Rightarrow \beta$. By Lemma 4, there is $\delta \in T'$ such that $\Phi \vdash_{T'} Z[\delta] \Rightarrow \beta$ and $\Phi \vdash_{T'} X \Rightarrow \delta$. Consequently $[\delta] \subseteq [Z[\circ],\beta]$, by (CUT), and $X \in [\delta]$. Clearly, we may assume $\delta \in R$. Let $\delta_1,\ldots,\delta_n \in R$ be all formulas obtained in this way, for a fixed $X \in U$ (we have $n \neq 0$, since U is non-total and closed). By γ_X we denote a formula from R which is $T'-$equivalent to $\delta_1 \wedge \cdots \wedge \delta_n$. By (15) and the definition of C, we obtain:

$$X \in [\gamma_X] \subseteq \bigcap\{[Z[\circ],\beta] \in \mathcal{B}(T) : U \subseteq [Z[\circ],\beta]\} = U\,.$$

Let α be the disjunction of all formulas γ_X, for $X \in U$ (the disjunction is nonempty, since $U \neq \emptyset$). By (15) and the above, we get $U \subseteq [\alpha] \subseteq U$, whence $U = [\alpha]$.

Consequently, the algebra is finite. We prove that its underlying lattice is distributive. It suffices to prove:

$$U \cap (V \vee_C W) \subseteq (U \cap V) \vee_C (U \cap W)\,, \tag{16}$$

for all closed sets U, V, W. This inclusion is true, if at least one of these sets is empty or total. So, assume that U, V, W are nontrivial. By the above, there exist $\alpha, \beta, \gamma \in R$ such that $U = [\alpha]$, $V = [\beta]$ and $W = [\gamma]$. Accordingly, $U \cap (V \vee_C W)$ equals $[\alpha] \cap ([\beta] \vee_C [\gamma])$. By (15), the latter equals $[\alpha \wedge (\beta \vee \gamma)]$. By (D) (restricted to $T'-$sequents) and (CUT), the latter is contained in $[(\alpha \wedge \beta) \vee (\alpha \wedge \gamma)]$. By (15), the latter set equals $(U \cap V) \vee_C (U \cap W)$. □

REMARKS. (1) For the $\vee-$free fragment of **FGL**, the corresponding lemma states that $\mathcal{M}(c(T))$ is a finite lattice-ordered RA (it is closed under finite meets, whence also under finite joins, although the latter are not represented in the language). The proof is different. We define an equivalence relation $\sim$ on $(T')^*$: $X \sim Y$ iff, for any $\alpha \in T'$, $X \in [\alpha]$ iff $Y \in [\alpha]$. By an analogue of Lemma 5 (see the remark), this relation has a finite index. We show that every set $[Z[\circ], \alpha] \in \mathcal{B}(T')$ is invariant with respect to $\sim$. Assume $X \in [Z[\circ], \alpha]$ and $X \sim Y$. Then, $\Phi \vdash_{T'} Z[X] \Rightarrow \alpha$. By Lemma 4, there exists $\delta \in T'$ such that $\Phi \vdash_{T'} Z[\delta] \Rightarrow \alpha$ and $\Phi \vdash_{T'} X \Rightarrow \delta$. Since $X \sim Y$, then $\Phi \vdash_{T'} Y \Rightarrow \delta$, whence $\Phi \vdash_{T'} Z[Y] \Rightarrow \alpha$, by (CUT). Therefore, $Y \in [Z[\circ], \alpha]$. Consequently, every closed set is invariant with respect to $\sim$, whence there are only finitely many closed sets. (2) A similar argument works for **GL**. Now $T' = T$, and one proves that $\mathcal{M}(T)$ is a finite RA.

Notice that in the proof of Lemma 7 we must show that (D) is valid in $\mathcal{M}(c(T))$. The proof employs the fact that $c(T)$ is closed under all operations, appearing in (D), and finite up to $c(T)-$equivalence. We show below that some other axioms can be handled in a similar way, but not arbitrary axioms, as e.g. the associative law for product (see comments under Lemma 4).

Lemma 8. *Let T be a nonempty set of formulas, closed under subformulas. Any valuation μ in $\mathcal{M}(T)$ such that $\mu(p) = [p]$, for all $p \in T$, satisfies $\mu(\alpha) = [\alpha]$, for all $\alpha \in T$. Also, for any $T-$sequent $X \Rightarrow \alpha$, this sequent is true in $(\mathcal{M}(T), \mu)$ if and only if $\Phi \vdash_T X \Rightarrow \alpha$ in* **DFGL**.

Proof. The first part can easily be proved by induction on α, using (13), (14), (15). We prove the second part. Assume that $X \Rightarrow \alpha$ is a $T-$sequent true in $(\mathcal{M}(T), \mu)$. Then, $\mu(F(X)) \subseteq \mu(\alpha)$. By the first part, $X \in \mu(F(X))$, whence $X \in \mu(\alpha) = [\alpha]$. Consequently, $\Phi \vdash_T X \Rightarrow \alpha$. Assume $\Phi \vdash_T X \Rightarrow \alpha$. We prove that $X \Rightarrow \alpha$ is true in $(\mathcal{M}(T), \mu)$, by induction on $T-$deductions. The axioms (Id), (D) and the assumptions from Φ, restricted to $T-$sequents, are of the form $\beta \Rightarrow \gamma$ such that $\beta, \gamma \in T$. By (CUT), $[\beta] \subseteq [\gamma]$, whence $\mu(\beta) \subseteq \mu(\gamma)$, by the first part. All rules of **FGL** preserve the truth in $(\mathcal{M}(T), \mu)$, since $\mathcal{M}(T)$ is a lattice-ordered RA. $\square$

We are ready to prove SFMP for **DFGL**.

Theorem 1. *Assume that $\Phi \vdash X \Rightarrow \alpha$ does not hold in* **DFGL**. *Then, there exist a finite distributive lattice-ordered RA $\mathcal{M}$ and an assignment μ such that all sequents from Φ are true but $X \Rightarrow \alpha$ is not true in $(\mathcal{M}, \mu)$.*

Proof. Let T be the set of all formulas appearing in Φ and $X \Rightarrow \alpha$. We denote $T' = c(s(T))$. Since $s(T)$ is finite, then $\mathcal{M}(T')$ is a finite distributive lattice-ordered RA, by Lemma 7. Clearly $\Phi \vdash_{T'} X \Rightarrow \alpha$ does not hold in **DFGL**. Define μ as in Lemma 8. By this lemma, all sequents from Φ are true in $(\mathcal{M}(T'), \mu)$ but $X \Rightarrow \alpha$ is not true. $\square$

REMARKS. (1) Since the class of distributive lattice-ordered RAs is closed under finite products, SFMP entails FEP for this class. (2) In a similar way, we prove FEP for the class of meet-semilattice ordered RAs and the class of RAs; use the above remarks and the fact that Lemma 8 is true for the $\vee-$free fragment of **FGL** and **GL** (for the latter, $T' = s(T)$).

Now, we can prove a version of the subformula property and a stronger interpolation lemma for each of these systems.

Corollary 1. *If* $\Phi \vdash X \Rightarrow \alpha$ *in* **DFGL**, *then* $\Phi \vdash_{T'} X \Rightarrow \alpha$ *in* **DFGL**, *where* T *is the set of formulas appearing in* Φ *and* $X \Rightarrow \alpha$ *and* $T' = c(s(T))$.

Proof. Assume $\Phi \vdash X \Rightarrow \alpha$. Let μ be defined as in Theorem 1. $\mathcal{M}(T')$ is a distributive lattice-ordered RA, and all sequents from Φ are true under μ. By the strong soundness of **DFGL** with respect to such algebras, $X \Rightarrow \alpha$ is true in $(\mathcal{M}(T'), \mu)$. By Lemma 8, $\Phi \vdash_{T'} X \Rightarrow \alpha$. □

REMARK. For the $\vee-$free fragment of **FGL**, an analogous lemma holds with the same proof except for dropping distribution. For **GL**, we take $T' = s(T)$.

Corollary 2. *If* $\Phi \vdash X[Y] \Rightarrow \alpha$, *then there is* $\delta \in T'$ *such that* $\Phi \vdash X[\delta] \Rightarrow \alpha$ *and* $\Phi \vdash Y \Rightarrow \delta$, *where* T' *is defined as in Corollary 1.*

Proof. We apply Corollary 1 and Lemma 4. □

REMARKS. (1) Again, the same is true for the $\vee-$free fragment of **FGL** and **GL** with appropriate modifications of T'. (2) Both corollaries are true for **FGL**, but Lemma 5 fails, whence T' cannot be reduced to a finite set. (3) We leave open whether Theorem 1 holds for **FGL**.

Corollaries 1 and 2 are essential in proof-theoretic decision procedures for **DFGL** and its variants. By these corollaries, every sequent $X \Rightarrow \alpha$, derivable from Φ, possesses a restricted deduction, in which any sequent $Y \Rightarrow \delta$ is of a very limited form: $Y = \beta$ or $Y = (\beta_1, \ldots, \beta_n)_o$, with δ, β, β_i coming from a finite set R, which depends on Φ and $X \Rightarrow \alpha$ (see a special case in [8]; further details will be provided in a forthcoming paper).

5 Extensions

The above results can also be obtained for different extensions of **DFGL**. In this section we consider several variants.

First, we add constants $\bot, \top$, interpreted as the lower bound and the upper bound of the lattice. The new axioms for them are:

$$(\bot\mathrm{L})\ X[\bot] \Rightarrow \alpha\,,\ \ (\top\mathrm{R})\ X \Rightarrow \top\,.$$

The resulting system is strongly complete with respect to bounded distributive lattice-ordered RAs. (One could introduce $\bot$ only and define $\top = (o/1)(\bot, \ldots, \bot)$, for an at least binary operation symbol o, but it is not expedient for our purposes.)

Lemma 4 remains true. The proof needs new cases for the new axioms (if T contains the corresponding additive constant). Consider $(\bot\mathrm{L})$ of the form

$X[Y] \Rightarrow \alpha$. If Y contains $\bot$, then we take $\delta = \bot$ as an interpolant of Y; otherwise, we take $\delta = \top$. Consider ($\top$R) of the form $X[Y] \Rightarrow \top$. Then, we take $\delta = \top$ as an interpolant of Y.

In the material of section 4 we assume that $c(T)$ always contains $\bot, \top$. Lemma 5 remains true. Lemma 6 goes without changes. In Lemma 7, $\mathcal{M}(c(T))$ is actually a finite distributive bounded lattice-ordered RA. Now, one easily proves that the lower bound is $C(\emptyset) = [\bot]$ and the upper bound is $(T')^* = [\top]$. Furthermore, every closed set equals $[\alpha]$, for some $\alpha \in T'$, whence one need not consider nontrivial sets. Lemma 8 remains true.

Therefore, Theorem 1 can be proved for **DFGL** with $\bot, \top$. It yields FEP for the class of bounded distributive lattice-ordered RAs. Corollaries 1 and 2 remain true. Dropping distribution, we obtain variants of these results (see remarks in section 4).

For some binary operation symbols o, one can introduce units 1_o, satisfying $o(1_o, a) = a = o(a, 1_o)$. Since 1_o is a new nullary operation, our system contains the axiom (1_oR): $()_{1_o} \Rightarrow 1_o$ and the rule:

$$(1_o\mathrm{L})\ \frac{X[()_{1_o}] \Rightarrow \alpha}{X[1_o] \Rightarrow \alpha}\,.$$

In the metalanguage we assume $(()_{1_o}, Y)_o = Y = (Y, ()_{1_o})_o$. This assumptions can be formalized by affixing some structural rules, allowing to replace every substructure Y by $(()_{1_o}, Y)_o$ and $(Y, ()_{1_o})_o$ and every substructure of the latter form by Y.

The resulting system is strongly complete with respect to distributive lattice-ordered RAs, which are unital with respect to some binary operations o. In Lemma 4 we assume that T also contains 1_o (if 1_o occurs in the system), which is an interpolant of $()_{1_o}$ in any context. In the material of section 4 we always assume that $c(T)$ contains 1_o (if 1_o occurs in the system). Then, the algebra $\mathcal{M}(c(T))$ is unital with respect to the appropriate operations o. First, we have $(1_o)_C = [1_o]$, as for an arbitrary nullary operation. Let U be a closed set. U equals the meet of all basic closed sets, containing U. If $X \in U$, then $(()_{1_o}, X)_o$, and consequently $(1_o, X)_o$, belongs to every basic closed set, containing U. By (CUT), $O([1_o], U)$ is contained in every basic closed set, containing U, whence $O_C([1_o], U) \subseteq U$. Let $[Z[\circ], \alpha]$ be a basic closed set, containing $O([1_o], U)$. Let $X \in U$. We have $(()_{1_o}, X)_o \in [Z[\circ], \alpha]$, whence $X \in [Z[\circ], \alpha]$. It yields $U \subseteq O_C([1_o], U)$, so $U = O_C([1_o], U)$. In a similar way one shows $U = O_C(U, [1_o])$. Now, all results of section 4 can be proved with appropriate modifications. Clearly, we can do both things: add $\bot, \top$ and units 1_o, still preserving our results.

For some at least binary operations o, one can add the exchange rules:

$$(o\mathrm{EXC})\ \frac{X[(Y_1, \ldots, Y_n)_o] \Rightarrow \alpha}{X[(Y_{\pi(1)}, \ldots, Y_{\pi(n)})_o] \Rightarrow \alpha}\,,$$

for any nontrivial permutation π of $\{1, \ldots, n\}$. The resulting system is strongly complete with respect to distributive lattice-ordered RAs, in which the distinguished operations are commutative. For a commutative operation o, all residuals o/i reduce to $o/1$, since e.g. $(o/2)(a_1, \ldots, a_n)$ equals $(o/1)(a_2, a_1, a_3, \ldots, a_n)$. For systems with exchange rules, Lemma 4 holds with the same proof (these rules do not affect interpolants). In algebras $\mathcal{M}(T)$, the distinguished operations are commutative, since every basic closed set, containing $O(U_1, \ldots, U_n)$, also contains $O(U_{\pi(1)}, \ldots, U_{\pi(n)})$. Then, all results of section 4 can be proved for systems with exchange rules and the corresponding classes of algebras. In particular, FEP holds for the class of (distributive lattice-ordered) RAs, in which some distinguished operations are commutative (also the lattice can be bounded, the algebras can be unital with respect to some operations).

For systems with $\bot, \top$, the weakening rules for some distinguished operations o can be added. They take the form:

$$(o\text{WEA}) \ \frac{X[Y] \Rightarrow \alpha}{X[(\ldots, Y, \ldots)_o] \Rightarrow \alpha}\,,$$

where Y is the i-th term in $(\ldots, Y, \ldots)_o$, and the remaining terms are arbitrary formula structures. The resulting system is strongly complete with respect to (distributive) bounded lattice-ordered RAs such that $o(a_1, \ldots, a_n) \leq a_i$ holds, for the distinguished $n-$ary operations o and $i = 1, \ldots, n$. This yields $1_o = \top$, if 1_o exists; then, we say that the algebra is *integral* with respect to o. In the proof of Lemma 4 (we assume that T contains $\top$), we take $\top$ as an interpolant of any new term, introduced by this rule; an interpolant of any substructure of the antecedent of the conclusion of (oWEA), containing $(\ldots, Y, \ldots)_o$ equals an interpolant of Y in the premise. All results of section 4 can easily be proved for systems with weakening rules and the corresponding classes of algebras. In particular, FEP holds for the class of bounded (distributive lattice-ordered) RAs, which are integral with respect to some distinguished operations.

Now, we consider **DFGL**, enriched with negation $\neg$, which together with $\wedge, \vee, \bot, \top$ satisfies the laws of boolean algebras. To **DFGL** with $\bot, \top$ we add new axioms:

$$(\neg 1)\ \alpha \wedge \neg\alpha \Rightarrow \bot\,,\ (\neg 2)\ \top \Rightarrow \alpha \vee \neg\alpha\,,$$

for all formulas α. Clearly the resulting system is strongly complete with respect to boolean RAs. We denote this system by **BGL**.

We introduce an auxiliary system **S**, which amounts to **DFGL** with $\bot, \top$ and a designated binary operation o'. We define $\neg\alpha = (o'/2)(\alpha, \bot)$ (the latter can be written as $\alpha \to \bot$) and admit axioms $(\neg 1)$, $(\neg 2)$ in **S**. Clearly **BGL** is a subsystem of **S** in the sense that the consequence relation of the former is contained in the consequence relation in the latter. Actually, $\Phi \vdash X \Rightarrow \alpha$ holds in **BGL** iff it holds in **S**, provided that all sequents in Φ and $X \Rightarrow \alpha$ are in the language of **BGL**. It follows from the fact that every boolean RA can be expanded to a model of **S**: just interpret $o'(a, b) = a \wedge b$, whence $(o'/2)(a, b) = \neg a \vee b$.

The reader may feel the introduction of **S** as an artificial move, the first one in this paper. We justify it as follows. In the construction of special models in the style of section 4, one must define all designated operations on closed sets.

We avoid the nontrivial problem of defining boolean negation $\neg U$, for a closed set U; we simply define it as $(O'/2)(U, \bot)$, as for an arbitrary operation o (here $\bot$ stands for the smallest closed set $C(\emptyset)$, which equals $[\bot]$ in $\mathcal{M}(c(T))$). We have to show that $\mathcal{M}(c(T))$ satisfies axioms $(\neg 1)$, $(\neg 2)$, but they will be treated similarly as (D).

Axioms $(\neg 1)$, $(\neg 2)$ are of the form $\beta \Rightarrow \gamma$, whence they cause no problem in the proof of Lemma 4. Accordingly, Lemma 4 is true for **S** (see above, for the treatment of $(\bot\mathrm{L})$, $(\top\mathrm{R})$). We define $c(T)$ as the closure of $T \cup \{\bot, \top\}$ under $\wedge, \vee, \neg$. Clearly Lemma 5 remains true for **S** (boolean algebras are locally finite). Precisely, every formula in $c(T)$ is $c(T)-$equivalent to a finite disjunction of finite conjunctions of formulas from $T \cup \{\bot, \top\}$ and their negations. Lemma 6 goes without changes. We add $\neg[\alpha] = [\neg\alpha]$ to equations (15); it holds by (14) and the fact that $\neg[\alpha] = (O'/2)([\alpha], [\bot])$. An analogue of Lemma 7 for **S** states that $\mathcal{M}(c(T))$ is a finite boolean RA, for any nonempty finite set T. The proof is as for **DFGL** with $\bot, \top$, and one must additionally prove that $\mathcal{M}(c(T))$ satisfies $U \cap \neg U \subseteq \bot$ and $\top \subseteq U \vee_C \neg U$, for any closed set U. We have already proven that $U = [\alpha]$, for some $\alpha \in c(T)$. So, the desired inclusions follow from the fact that $[\alpha \wedge \neg\alpha] \subseteq [\bot]$ and $[\top] \subseteq [\alpha \vee \neg\alpha]$. Lemma 8 goes without changes. This yields an analogue of Theorem 1, i.e. SFMP for **S**. Since **S** is conservative over **BGL**, we obtain SFMP for **BGL**, and consequently, FEP for the class of boolean RAs. Corollaries 1 and 2 for **BGL** remain true.

Instead of boolean algebras we may consider Heyting algebras, i.e. bounded lattices with binary operations $\vee, \wedge, \rightarrow$ such that $\rightarrow$ is the residual operation for $\wedge$. A Heyting RA is an RA with $\wedge, \vee, \rightarrow, \bot, \top$ such that the reduct to the latter operations is a Heyting algebra. **HGL** is obtained from **FGL** with $\bot, \top$ and (CUT) by affixing two rules:

$$\text{(R1)}\ \frac{\alpha \wedge \beta \Rightarrow \gamma}{\beta \Rightarrow \alpha \rightarrow \gamma}\,,\ \text{(R2)}\ \frac{\beta \Rightarrow \alpha \rightarrow \gamma}{\alpha \wedge \beta \Rightarrow \gamma}\,.$$

for any formulas α, β, γ. Clearly **HGL** is strongly complete with respect to the class of Heyting RAs. Notice that (D) is provable in **HGL**, but the proof uses formulas with $\rightarrow$.

We introduce an auxiliary system **SH**. It amounts to **DFGL** with $\bot, \top$, which admits the following axioms, for some designated binary operation symbol o':

$$\text{(H1)}\ o'(\alpha, \beta) \Rightarrow \alpha \wedge \beta\,,\ \text{(H2)}\ \alpha \wedge \beta \Rightarrow o'(\alpha, \beta)\,.$$

We write $\alpha \rightarrow \beta$ for $(o'/2)(\alpha, \beta)$. (R1) and (R2) are derivable rules of **SH**, whence the consequence relation of **HGL** is contained in that of **SH**. The latter is conservative over the former, since every Heyting RA can be expanded to a model of **SH**; just define $o'(a, b) = a \wedge b$.

For **SH** we proceed as above. Lemma 4 can be proved as for **DFGL** with $\bot, \top$. We define $c(T)$ as the closure of $T \cup \{\bot, \top\}$ under $\vee, \wedge, o'$. Since every formula from $c(T)$ is $c(T)-$equivalent to a finite disjunction of finite conjunctions of formulas from $T \cup \{\bot, \top\}$, then Lemma 5 for **SH** is true. Lemma 6 goes without changes. An analogue of Lemma 7 states that $\mathcal{M}(c(T))$ is a finite

Heyting RA, for any nonempty finite set T. This algebra satisfies (H1) and (H2), by an argument similar to the above for ($\neg$1), ($\neg$2). Lemma 8 goes without changes. It yields Theorem 1 for **SH**, i.e. SFMP for **SH**. Since **SH** is conservative over **HGL**, then we obtain SFMP for **HGL**, and consequently, FEP for the class of Heyting RAs.

At the end, we briefly discuss double RAs and their logics.

For any boolean RA and any designated operation o, one can define its De Morgan dual o^d:

$$o^d(a_1,\ldots,a_n) = \neg o(\neg a_1,\ldots,\neg a_n)\,, \tag{17}$$

for all elements $a_1,\ldots,a_n$. One easily shows that o^d admits residual operations o^d/i, for $i = 1,\ldots,n$, with respect to the reverse ordering $\geq$ such that $o^d/i = (o/i)^d$.

A unary operation o can be interpreted as the possibility operation $\Diamond$. Since $\bot \leq (o/1)(\bot)$, then $o(\bot) \leq \bot$, by (3), whence $o(\bot) = \bot$ holds in any boolean RA. The dual o^d can be interpreted as the classical $\Box$. In any boolean RA, $o(\neg\top) \leq \neg\top$, then $\top \leq \neg o(\neg\top)$, whence $o^d(\top) = \top$. For formulas α in the language of modal logics, one can show that $\top \Rightarrow \alpha$ is valid in boolean RAs if and only if α is a theorem of modal logic **K**. Therefore, **K** can be faithfully interpreted in **BGL**. This interpretation can be extended to other modal logics, e.g. **T**, **S4**. Instead of new axioms, added to **BGL**, one can add new structural rules for $(-)_o$. Therefore, our methods provide new proofs of FMP for some modal logics. We defer a detailed discussion to a forthcoming paper.

A double RA is an algebra $\mathcal{M}$ of signature $(\sigma_r, (\sigma')_r)$ and ordering $\leq$ whose basic algebra is of signature (σ, σ'), the reduct of $\mathcal{M}$ of signature σ_r is an RA with respect to $\leq$, and the reduct of $\mathcal{M}$ of signature $(\sigma')_r$ is an RA with respect to $\geq$. By o we denote any operation connected with σ and by ω any operation connected with σ'. Hence any non-nullary operation ω, satisfies the following variant of (2):

$$b \leq \omega(a_1,\ldots,a_n) \text{ iff } (\omega/i)(a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_n) \leq a_i \tag{18}$$

for all $i = 1,\ldots,n$ and all elements $a_1,\ldots,a_n,b$.

The order formulas $\alpha \leq \beta$ valid in double RAs (of a fixed signature) are derivable in a variant of **GL**, denoted by **GLd**. Its algebraic form enriches the algebraic form of **GL** by rules corresponding to (18). The sequential form can be designed as well, but we do not employ it here.

Kurtonina and Moortgat [17] consider a particular system of this kind with one binary operation o and one binary operation ω (they call it the Lambek-Grishin calculus). They prove that this system is complete with respect to Kripke structures (S, R_o, R_ω) such that S is a nonempty set and $R_o, R_\omega \subseteq S^3$. The operations O, Ω on $P(S)$ are defined as follows:

$$O(U,V) = \{x \in S : (\exists y \in U)(\exists z \in V)R_o(x,y,z)\}\,, \tag{19}$$

$$\Omega(U,V) = \{x \in S : (\forall y,z \in S)(\text{ if } R_\omega(x,y,z) \text{ then } y \in U \text{ or } z \in V)\}\,, \tag{20}$$

for $U, V \subseteq S$. Clearly Ω is the De Morgan dual of some operation, defined from R_ω according to (19). Since O (resp. Ω) is distributive under infinite joins (resp. meets) in both arguments, then their residual operations exist and are uniquely determined; the residuals of Ω are De Morgan duals of those of O. $P(S)$ with $\subseteq$ and the operations $O, O/1, O/2$ and $\Omega, \Omega/1, \Omega/2$ is a double RA. The completeness proof in [17] yields, actually, the strong completeness of **GLd** (with o, ω and their residuals) with respect to double RA's of this particular kind. It can easily be adapted to arbitrary signatures with natural modifications of (19), (20).

Accordingly, **GLd** is strongly complete with respect to the class of double RAs in which operations from $(\sigma')_r$ arise by (17) from some operations satisfying (2). This yields a faithful interpretation of the consequence relation of **GLd** in the consequence relation of **BGL**. The algebraic version of this result is that every double RA can be embedded into a boolean RA. It follows that **GLd** possesses SFMP, and consequently, FEP holds for the class of double RAs.

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